

RICE UNIVERSITY

**Minimizers of the vector-valued coarea formula.**

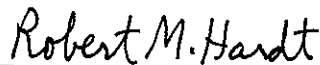
by

**Colin Daniel Carroll**

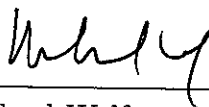
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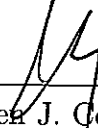
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# **Abstract**

Minimizers of the vector-valued coarea formula

by

Colin Daniel Carroll

The vector-valued coarea formula provides a relationship between the integral of the Jacobian of a map from high dimensions down to low dimensions with the integral over the measure of the fibers of this map. We explore minimizers of this functional, proving existence using both a variational approach and an approach with currents. Additionally, we consider what properties these minimizers will have and provide examples. Finally, this problem is considered in metric spaces, where a third existence proof is given.

# Acknowledgments

First and foremost, the author would like to acknowledge Dr. Robert Hardt for much guidance, unbounded patience, and an always open door. The author is also grateful for the opportunity to study at such an active and encouraging department as the one that has been chaired by Dr. Michael Wolf and Dr. Brendan Hassett over the past five years. The author would like to thank postdoctoral faculty Rolf Ryham, as well as his fellow graduate students Casey Douglas, Renée Laverdière, Thomas McGaffey, Ryan Scott, and Zhi Zhang for providing so many helpful discussions. Thanks is also due to Karin Knudson for her understanding and support. Finally, none of this would have been possible without the phenomenal math department at Williams College – especially the enthusiasm of Prof. Frank Morgan and Prof. Satayan Devadoss – as well as the encouragement of the author’s friends and family.

*To Anne, Robert, Emily and Tim –*

*Dedications often sound really phony.*

*Any expression of affection can look phony.*

*My family has loved me and given me everything*

*I’ve needed or wanted for 27 years.*

*I hope they know how I appreciate each of them.*

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# Chapter 1

## Introduction.

In this chapter I give a broad outline of variational problems, describe my main results, and give an outline of the remainder of the thesis.

### 1.1 The calculus of variations.

Broadly, we will be working in the field of the *calculus of variations*: the study of minimizing certain a certain “energy”. Explicitly, if  $\Omega \subset \mathbb{R}^m$  is bounded and open with Lipschitz boundary, and the *Lagrangian*  $L : \mathbb{R}^m \times \mathbb{R} \times \overline{\Omega} \rightarrow \mathbb{R}$  is a smooth function, then this “energy” may have the form

$$I[u] := \int_{\Omega} L(Du(x), u(x), x) \, dx. \quad (1.1)$$

In the sequel, we will write  $L = L(p, z, x) = L(p_1, \dots, p_m, z, x_1, \dots, x_m)$ , so that  $D_p L = (L_{p_1}, \dots, L_{p_m})$ .

Historically, the first motivation for the development of this theory was the *brachistochrone* problem – the problem of finding the curve between two points so that a



ball moving by gravity travels between the points the fastest. The solution to these problems will be a minimizing curve (or function), which contrasts with ordinary calculus, whose solutions are minimizing points.

Euler developed this theory more systematically, in order to find minimizers of more general functionals as in Equation 1.1. Among the more well studied problems was to minimize the *area functional*,  $A$ , which gives the area of a parameterized surface  $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ :

$$A[u] := \int_{\Omega} |u_{x_1} \times u_{x_2}| \, dx, \quad (1.2)$$

subject to some boundary conditions. The boundary conditions may be specified via some given  $g : \partial\Omega \rightarrow \mathbb{R}^3$ , so that a by a solution to Equation 1.2 we mean that  $u|_{\partial\Omega} = g$  and

$$A[u] = \inf_{w|_{\partial\Omega}=g} A[w].$$

We discuss the area functional more carefully in Chapter 2.

One of the main tools in studying such problems – the Euler-Lagrange equations – gives a partial differential equation which must be satisfied by any solution to such a problem (see Section 3.4). The intuition is that if  $u$  minimizes Equation 1.1, then for any smooth function  $\phi$ , compactly supported in  $\Omega$ , the *real valued* function

$$i(t) := I[u + t\phi] = \int_{\Omega} L(Du + tD\phi, u + t\phi, x) \, dx$$

has a critical point at  $t = 0$ . This translates the question of minimizing a functional whose domain does not have a finite basis to a question of minimizing a real valued function, so that traditional tools of calculus may be applied. Specifically, since  $t = 0$

is a critical point, computing the derivative of  $i$  with respect to  $t$  and then setting  $t = 0$  yields

$$\int_{\Omega} L_{\mathbf{p}}(Du, u, x) \cdot \nabla \phi + L_z(Du, u, x) \phi \, dx = 0, \quad (1.3)$$

for all test functions  $\phi$ . One may then argue that

$$- \operatorname{div}(L_{\mathbf{p}}(Du, u, x)) + L_z(Du, u, x) = 0.$$

Equation 1.3 is the weak Euler-Lagrange equation associated with  $I[\cdot]$ . See [Eva97],[GF63] for a more thorough discussion of this calculation and its applications to the calculus of variations.

Equation 1.3 motivates the use of Sobolev functions (see Chapter 3) in the study of the calculus of variations. Specifically, one may think of Sobolev spaces as the space of functions so that the Euler-Lagrange equation is weakly defined. The space of Sobolev functions also has the property that the unit ball is weakly compact, which allows for the following strategy in solving a variational problem (I will use this argument in Chapters 3, 5, and 6), called the *direct method in the calculus of variations*.

Suppose  $I[\cdot]$  is a functional as above,  $\Omega \subset \mathbb{R}^m$ , and  $g : \partial\Omega \rightarrow \mathbb{R}$  is given. Further, let  $\mathcal{A}$  be our set of admissible functions: for admission to  $\mathcal{A}$ , we would typically require that a function  $u : \Omega \rightarrow \mathbb{R}$  has  $u|_{\partial\Omega} = g$ , plus some other amount of regularity—perhaps  $u$  must be Sobolev, or Lipschitz. Then we must show:

1. the set  $\mathcal{A}$  of admissible functions is non-empty;

2. the number  $M := \inf_{u \in \mathcal{A}} I[u]$  exists;
3. given a sequence  $\{u_j\} \subset \mathcal{A}$  with  $\lim_{j \rightarrow \infty} I[u_j] = M$ , there is a  $u \in \mathcal{A}$  and a weakly convergent subsequence (not relabeled) with  $u_j \rightharpoonup u$ ; and
4. the energy of the target function attains the infimum:  $I[u] = M$ .

We point out that Property 3 often follows after showing some weak compactness of the unit ball, and that candidate minimizers lie inside some bounded region (or that the infimum of  $I[\cdot]$  applied candidate minimizers in that region is no larger than  $M$ ). Demonstrating Property 4 will often require demonstrating the lower semicontinuity of  $I[\cdot]$ . That is, showing that

$$I[\liminf_{j \rightarrow \infty} u_j] \leq \liminf_{j \rightarrow \infty} I[u_j].$$

One then may conclude that there exists a minimizer of  $I[\cdot]$  in  $\mathcal{A}$ . Note that Properties 1-4 may be easier or harder to show depending on conditions imposed on the functional  $I[\cdot]$ , the domain  $\Omega$ , the boundary data  $g$  and the admissible functions  $\mathcal{A}$ .

## 1.2 Geometric measure theory.

Another tool that will prove useful in this thesis is geometric measure theory, which was developed in part to study area-minimizing surfaces. See for example [Fed69], [KP08], [FX03], or [Mor09] for more detail. In particular, geometric measure theory uses *Hausdorff measure* to investigate low dimensional sets. We define Hausdorff

measure by first letting the diameter of a set  $S \subset \mathbb{R}^n$  be defined by

$$\text{diam}(S) := \sup\{|x - y| : x, y \in S\}.$$

Then the  $m$ -dimensional Hausdorff measure of  $\Omega \subset \mathbb{R}^n$  is defined by

$$\mathcal{H}^m(\Omega) := \liminf_{\delta \rightarrow 0} \left\{ \sum_{j=1}^{\infty} \frac{\alpha(m)}{2^m} \text{diam}(S_j)^m : \Omega \subset \cup S_j, \text{diam } S_j \leq \delta \right\}. \quad (1.4)$$

Here  $\alpha(m)$  is the volume of a unit ball in  $\mathbb{R}^m$ . See [Fal86] for a more thorough introduction to Hausdorff measure.

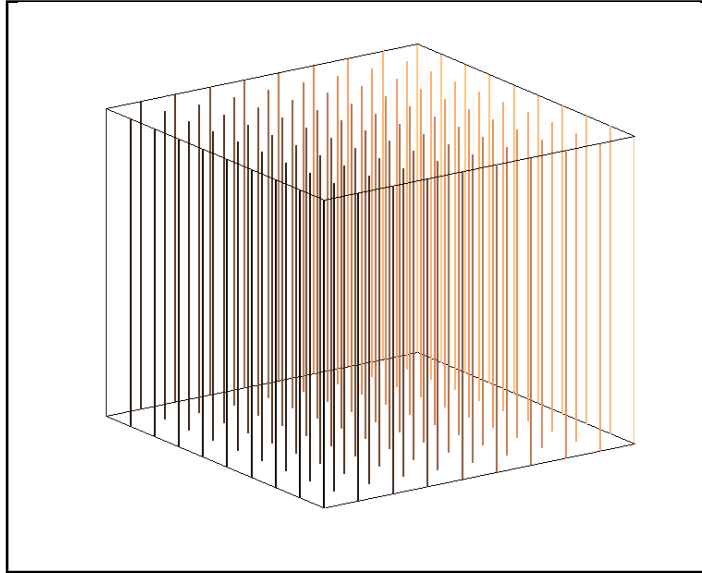
This means that instead of looking at the problem of minimizing Equation 1.2, or even finding  $(n - 1)$ -dimensional sets in  $\mathbb{R}^n$  with prescribed  $(n - 2)$ -dimensional boundary, one may ask: for  $1 \leq k < n$ , what  $k$ -dimensional set in  $\mathbb{R}^n$  with prescribed  $(k - 1)$ -dimensional boundary has least  $k$ -dimensional area?

Where Hausdorff measure is the ruler of geometric measure theory, the objects to measure are *currents*, which are continuous linear functionals on the space of smooth, compactly supported differential forms. Currents provide a pleasant geometric framework in which to operate, especially when minimizing a geometric quantity. Also, certain spaces of currents have nice compactness properties, which motivates their use in geometric variational problems which will involve the direct method. Chapter 5 provides more background on currents and motivation for their use in this research.

### 1.3 An overview of the problem.

We briefly state the problem here. See Chapter 2 for more detail on previous work.

The general problem in this thesis concerns minimizing the Jacobian integral of a



**Figure 1.1:** Fibers of the projection map from the cube to the square. The coarea formula asserts that the integral of the length of these fibers over the square is equal to the integral of the Jacobian over the cube.

map from higher dimensions into lower dimensions, a nonlinear problem. In case the map is Lipschitz continuous, Federer [Fed59] established the *coarea formula*, which expresses a relationship between this integral and the measure of the fibers of the map. This allows the question of minimizers of the Jacobian integral to be looked at from two directions, one analytic and one geometric:

1. Which maps minimize the integral of the Jacobian?
2. Which maps minimize the integral over the measures of their fibers *and* satisfy the coarea formula?

More specifically, if  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is  $C^1$  (a requirement which may be weakened) with  $m \geq n$ , the  $n$ -dimensional Jacobian is given by

$$|J_n f| = \sqrt{\det(Df \cdot Df^T)},$$

where  $Df$  is the  $n \times m$  matrix of partial derivatives. One may then ask to minimize the quantity

$$\int_{\Omega} |J_n f(x)| \, d\mathcal{L}^m(x)$$

over a given region  $\Omega \subset \mathbb{R}^m$  when the boundary data is specified. We will also explain what it means to minimize coarea *locally* in Remark 3.9, and provide an example of a map which does so in Section 4.3. In certain situations (for example, Federer assumed  $f$  to be a Lipschitz function), we have the coarea formula, which relates this integral to the integral over the fibers of  $f$ :

$$\int_{\Omega} |J_n f(x)| \, d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(\Omega \cap f^{-1}(y)) \, d\mathcal{L}^n(y).$$

Then by a “minimizer of the vector-valued coarea formula”, as in the title of this thesis, we mean any admissible function minimizing *either* side of this equality (in particular, when the equality holds).

## 1.4 Main results, outline.

In Chapter 2, I provide a description of previous work on the problem, as well as some theorems and notation which will be useful in the sequel.

Chapter 3 provides some progress towards a solution using classical analysis. We

use a generalization of the space of bounded variation, denoted  $BnV$ , to define a set

$$\mathcal{F}_C^{m,n}(\Omega, \mathbb{R}^n) = \{\mathbf{f} \in W^{1, \frac{mn}{m+1}} \cap BnV(\Omega; \mathbb{R}^n) : \|\mathbf{f}\|_{W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)} < C\}$$

of admissible functions for the boundary value problem.

Then the direct method in the calculus of variations provides a proof of Theorem 3.10:

**Theorem 3.10** (Existence for 2-dimensional range). *Let  $\Omega$  be an open and bounded subset of  $\mathbb{R}^m$  with Lipschitz boundary,  $C > 0$  be constant, and  $\mathbf{g} \in \mathcal{F}_C^{m,2}(\Omega, \mathbb{R}^2)$  be given. Then there exists a function  $\mathbf{u} \in \mathcal{F}_C^{m,2}(\Omega, \mathbb{R}^2)$  with  $\mathbf{u}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$  so that  $|J\mathbf{u}|(\Omega) \leq |J\mathbf{w}|(\Omega)$  for all  $\mathbf{w} \in \mathcal{F}_C^{m,2}(\Omega, \mathbb{R}^2)$  with  $\mathbf{w}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$ .*

This is followed by a discussion of the necessity of the  $W^{1, \frac{3}{2}}$  bound on admissible functions. There is also a discussion of the Euler-Lagrange equations, as well as the possibility of solving this equation using a gradient flow. The chapter concludes with another explicit example of a coarea minimizer, which illustrates the lack of uniqueness of minimizers.

Chapter 4 discusses properties of minimizing functions. As a means of generating coarea minimizers, we provide Theorem 4.4:

**Theorem 4.4.** *Suppose there exists a smooth non-constant horizontally homothetic harmonic morphism  $u : \Omega \rightarrow \mathbb{R}^n$  such that  $u|_{\partial\Omega} = g$ . Then  $u$  minimizes the coarea among all smooth functions with  $u|_{\partial\Omega} = g$ .*

This, combined with a classical result of Jacobi which characterizes all harmonic morphisms, provides an ample stable of examples. Also, once one has a coarea min-

imizer (not necessarily a harmonic morphism), we may generate new examples via Theorem 4.6

**Theorem 4.6.** *Suppose  $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$  has  $|JF(a)| \equiv 1$  for  $a \in \mathbb{R}^n$ ,  $\Omega \subset \mathbb{R}^m$  is open and bounded with Lipschitz boundary, and  $u \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  is a coarea minimizer. Then  $F \circ u : \Omega \rightarrow \mathbb{R}^n$  minimizes  $|Jg|(\Omega)$  among all functions  $g \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  with  $g|_{\partial\Omega} = (F \circ u)|_{\partial\Omega}$ .*

We also compute a number of explicit examples of coarea minimizers, constructed from harmonic morphisms (one of which is the Hopf fibration), as critical points of the Euler-Lagrange equations associated with the functional, or as a certain class of ruled surfaces.

A very different approach is then introduced in Chapter 5, viewing the problem from the perspective of slices of rectifiable currents. This is motivated by the observation that the integral of the mass of the slices of a rectifiable set by a Lipschitz function is equal to the integral of the  $m - n$ -dimensional measure of the intersection of inverse images of the Lipschitz function with the rectifiable set, which is expressed in Equation 5.5:

$$\int_{\mathbb{R}^n} \mathbb{M}(\langle \tau(\Omega, \theta, \vec{\Omega}), f, y \rangle) d\mathcal{L}^n(y) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f_j^{-1}(y) \cap \Omega) d\mathcal{L}^n(y). \quad (1.5)$$

Background and notation is given, building up to Theorem 5.11, which states

**Theorem 5.11** (Existence). *Suppose  $\Omega \subset \mathbb{R}^m$  is rectifiable and bounded,  $\ell \geq 0$ ,  $h \in Lip_\ell(\overline{\Omega}; \mathbb{R}^n)$ . Then there exists an  $f \in Lip_\ell(\overline{\Omega}, \mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f^{-1}(y) \cap \Omega) d\mathcal{H}^n(y) \leq \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(g^{-1}(y) \cap \Omega) d\mathcal{H}^n(y)$$



for all  $g \in \text{Lip}_\ell(\overline{\Omega}, \mathbb{R}^n)$  with

$$\partial \langle \tau(\Omega, \theta, \vec{\Omega}), g, y \rangle = \partial \langle \tau(\Omega, \theta, \vec{\Omega}), h, y \rangle,$$

where  $\vec{\Omega}$  is an orienting vector field for  $\Omega$ .

This theorem generalizes Theorem 3.10 to arbitrary dimensions  $m > n > 1$ , at the cost of requiring more regularity in the set of admissible functions. Hence when both are applicable, the minimizers of Theorem 5.11 will necessarily have at least the same coarea as those of Theorem 3.10.

The final Chapter 6 discusses minimizers of the coarea formula in the context of recent work by De Pauw and Hardt [DPH12] on currents in metric spaces. The language and theory of the subject are reviewed, which leads up to a final existence proof, Theorem 6.11:

**Theorem 6.11.** *Suppose  $X$  is a metric space,  $T = \llbracket \gamma, \{A_k\}, g \rrbracket \in \mathcal{R}_m(X; \mathbb{Z})$  is an  $m$ -dimensional rectifiable  $\mathbb{Z}$ -chain with  $\sum_{k=1}^{\infty} \int_{\gamma(\partial A_k)} |g \circ \gamma_k^{-1}| d\mathcal{H}^{m-1} < \infty$ , and  $h \in \text{Lip}_\ell(X, \mathbb{R}^n)$  with  $m > n$ . Then for any compact  $K \subset \gamma(\bigcup_k A_k)$ , there exists an  $\hat{f} \in \text{Lip}_\ell(X, \mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} \hat{\mathbb{M}}_K(\langle T, \hat{f}, y \rangle) d\mathcal{L}^n(y) \leq \int_{\mathbb{R}^n} \hat{\mathbb{M}}_K(\langle T, f, y \rangle) d\mathcal{L}^n(y)$$

for all  $f \in \text{Lip}_\ell(X, \mathbb{R}^n)$  with

$$\partial \langle T, f, y \rangle = \partial \langle T, h, y \rangle.$$

# Chapter 2

## Background.

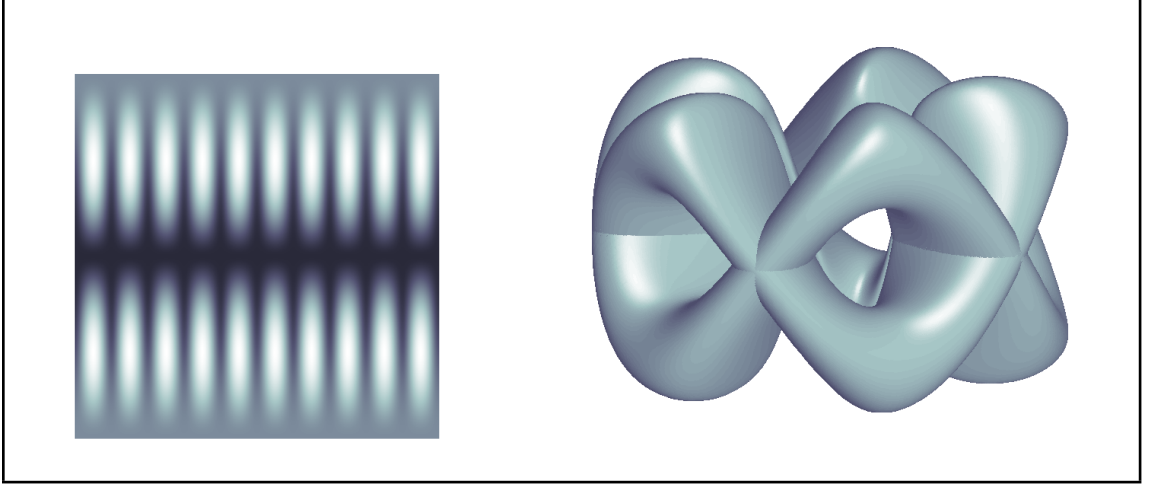
### 2.1 Early work.

It was not until 1959 that the coarea formula was first established by Herbert Federer for Lipschitz functions [Fed59]. Following the proof, he remarks that

...undoubtedly it would be possible to develop (for  $m \geq k$ ) a theory of “coarea” dual to the existing (for  $m \leq k$ ) theory of Lebesgue area.

Such a developed theory does not exist for a general target dimension, though there has been some work done in this direction, particularly when the target space is  $\mathbb{R}$  ([All08],[BDGG69],[CL97],[Mir65],[Par77]).

Later, Federer [Fed69] expands on his development of the coarea formula in the more general sense of studying the integral of the Jacobian, which makes the above comments more clear. In particular, for a Lipschitz function  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  with  $m \leq n$ , we have the classical area formula, relating the integral of the Jacobian to



**Figure 2.1:** The area formula relates the surface area of a parametrized surface, plotted on the right, with the integral of the Jacobian of that parameterization, plotted as an intensity on the left.

the  $m$ -dimensional measure of the image:

$$\int_{\Omega} |J_m f(x)| d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^0(\Omega \cap f^{-1}(y)) d\mathcal{H}^m(y), \quad (2.1)$$

where  $\mathcal{H}^0$  denotes the counting measure. This makes rigorous the interpretation of the Jacobian as a “stretching factor”. See figure 2.1.

Proceeding analogously when  $m \geq n$  (in particular,  $m > n$ ), we find that the intuition of the Jacobian measuring how much the image of a unit  $m$ -cube “stretches” still holds. In particular, we have

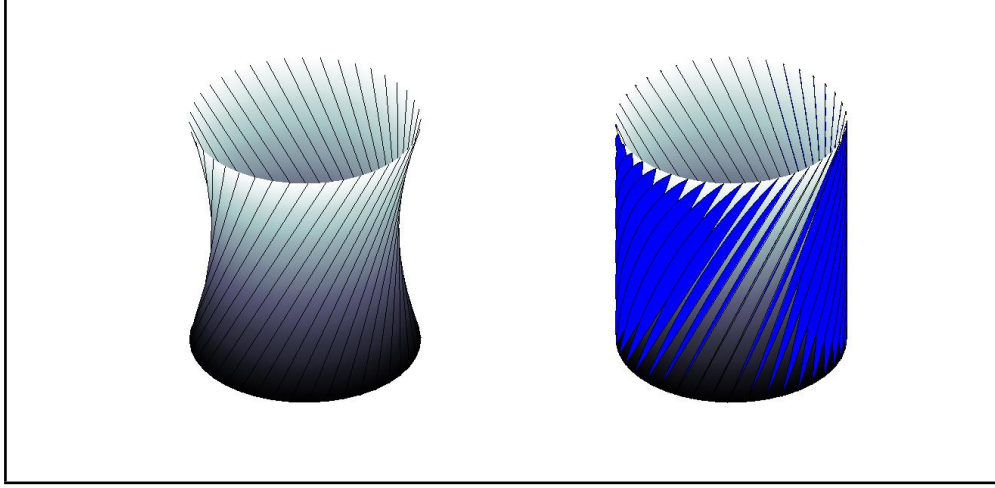
$$\int_{\Omega} |J_n f(x)| d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(\Omega \cap f^{-1}(y)) d\mathcal{L}^n(y). \quad (2.2)$$

It may be tempting to think of the righthand side of eq. 2.2 as the  $m$ -dimensional volume of  $\Omega$ . However, this is *incorrect* in most cases. To get a feel for the interaction between this stretching and volume, it may be helpful to consider three examples of

maps from the cube,  $C = [0, 1]^3 \subset \mathbb{R}^3$ , to  $\mathbb{R}^2$ :

1. If  $\pi : C \rightarrow \mathbb{R}^2$  is the projection map,  $\pi(x, y, z) = (x, y)$ , then  $|J\pi(\mathbf{x})| \equiv 1$ , each of the fibers over the unit square has length one, and we *do* recover the volume of the cube. See Figure 1.1.
2. However, if  $f : C \rightarrow \mathbb{R}^2$  is instead defined by  $f(x, y, z) = (\alpha x, \beta y)$ , then  $|Jf(\mathbf{x})| \equiv |\alpha\beta|$ , so  $\int_C |Jf(\mathbf{x})| d\mathcal{L}^3(\mathbf{x}) = |\alpha\beta|$ . As it must, this matches with the righthand side of the equation, since each fiber again has length one, but only over the rectangle  $[0, \alpha] \times [0, \beta]$ . Notice that we no longer recover the volume of the cube.
3. As an extreme example, Kaufman [Kau79] constructed an example using Cantor sets of a  $C^1$  map of the cube onto the square that has rank at most one everywhere. Hence, the coarea of the map is zero.

Each of the examples above also happen to minimize the coarea: the first two since the fibers are straight lines (cf. Theorem 4.7), and the last since the coarea is zero. Figure 2.2 provides another example of a minimizer (which has both non-zero coarea, and which is not a projection map), that will be described in detail in Section 3.6.



**Figure 2.2:** The inverse image of a coarea minimizer from the solid cylinder to the disk:

The left is the inverse image of an interior circle, and the right is the inverse image of the boundary of the disk. See Section 3.6 for details on this example.

## 2.2 Recent work.

In the case that  $n = 1$ , the left hand side of Eq. 2.2 reduces to the *total variation* of  $f$ :

$$TV(f) = \int_{\Omega} |\nabla f| \, d\mathcal{L}^m(x), \quad (2.3)$$

though this “energy” is often defined in a much more general sense. In particular, one looks at functions in the space  $BV(\Omega)$ , which are those functions whose distributional derivative is a finite Radon measure,  $Df$ . Specifically, we say  $f \in BV(\Omega)$  if there exists a finite Radon measure  $Df$  so that for each  $\varphi \in C_C^1(\Omega; \mathbb{R}^n)$ ,

$$-\int_{\Omega} \langle \varphi, Df \rangle = \int_{\Omega} f \, \operatorname{div} \varphi \, d\mathcal{L}^m.$$

Then, rather than defining the total variation as in Eq. 2.3, the total variation (i.e., coarea, i.e., integral of the Jacobian) of a function  $f$  is

$$TV(f) := \sup \left\{ - \int_{\Omega} \langle \varphi, Df \rangle : \varphi \in C_C^1(\Omega; \mathbb{R}^m), \|\varphi(x)\|_{L^\infty(\Omega)} \leq 1 \right\}, \quad (2.4)$$

which one can check is equal to Eq. 2.3 in the case that  $f \in C^1(\Omega)$ .

The total variational functional is of great interest in image processing [CL97], where the tools of geometric measure theory have been applied to establish some regularity of the levels [All08] for total variation minimizers. In particular, given a possibly degraded image  $u_0 : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ , where we interpret  $u_0(x, y)$  as the (greyscale) intensity of  $u_0$  at the pixel  $(x, y)$ , and an  $\epsilon > 0$ , one seeks to find an image  $u$  which minimizes a quantity similar to

$$F_\epsilon(u) := \int_{\Omega} |u - u_0|^2 d\mathcal{L}^2(x) + \epsilon TV(u).$$

There has been some research into defining a suitable analogue to  $BV(\Omega)$  for functions whose range is  $\mathbb{R}^n$  for  $n > 1$ , but the nonlinearity of the Jacobian makes such a definition difficult. In particular, Jerrard and Sonner [JS02], define *functions of bounded  $n$ -variation*, as functions  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  where the distributional determinant  $\det(f_{x_{\alpha_1}}^1, \dots, f_{x_{\alpha_n}}^n)$  is a measure for all choices of  $1 \leq \alpha_1 \leq \dots \leq \alpha_n \leq m$ . If  $n = 1$ , this is the classical space of bounded variation. But for  $n > 1$ , this is not even a linear space, much less a Banach space. They also give a weak version of the coarea formula and chain rule for functions in  $BnV$ , and describe continuity properties. Of particular use will be the lower semicontinuity of the weak coarea formula for these functions, Lemma 3.5. We will describe this framework more carefully in Chapter 3.

The above work concerns the analytic side of the problem, which is to say: in what contexts is the integral of the Jacobian (weakly) defined? We are also interested in work that has been done on the geometric end: that is, when does the coarea formula actually hold. Malý, Swanson and Ziemer [MSZ02] weakened Federer's requirement of Lipschitz functions to precise representatives of functions in  $W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$ , where  $1 \leq m < n$ , and  $p > m$ , as well as establishing the formula when the gradient of  $f$  is in a Lorentz space.

Recall that  $\tilde{f}$  is a *precise representative* of  $f \in L^1_{loc}(\Omega)$  if

$$\tilde{f}(x) := \lim_{r \rightarrow 0} \int_{B(x,r)} f(y) dy,$$

whenever this limit is defined.

Of use to us here will be the following theorem:

**Theorem 2.1** ([MSZ02]). *Suppose that  $1 \leq n \leq m$ ,  $\Omega \subset \mathbb{R}^m$  is open, and that  $f \in W^{1,p}_{loc}(\Omega; \mathbb{R}^n)$  is precisely represented, where  $p > n$ . Then  $f^{-1}(y)$  is countably  $\mathcal{H}^{m-n}$  rectifiable for almost all  $y \in \mathbb{R}^n$  and the coarea formula holds for all measurable sets  $E \subset \Omega$ .*

We also follow some development of the theory of functions of least gradient, i.e., total variation minimizers. Parks [Par77] first looked at existence for these functions. He used results from Miranda [Mir65] to show that for a strictly convex domain with boundary values satisfying a bounded slope condition, a unique Lipschitz solution exists. Specifically, let  $[\![\Omega]\!]$  denote the rectifiable current induced by the  $m$ -rectifiable set  $\Omega \subset \mathbb{R}^m$  (we will later denote this  $\tau(\Omega, \theta, \vec{\Omega})$ , with  $\theta \equiv 1$  and  $\vec{\Omega}$  the standard orientation of  $\mathbb{R}^m$ ), then

**Lemma 2.2** ([Par77]). *Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and uniformly convex,  $\Gamma_0 = \partial[\![\Omega]\!]$ , and  $\Gamma := \text{supp } \Gamma_0$ . Fix a  $Z_0 \subset \mathbb{R}^n$  to be open and bounded with  $Z_0 \cap \Gamma \neq \emptyset$  and  $\Gamma \setminus Z_0 \neq \emptyset$ . Also, for  $\phi : \Gamma \rightarrow \mathbb{R}$ , let  $\mathcal{B}(\phi) := \{u \in \text{Lip}(\overline{\Omega}) : u|_{\Gamma} = \phi\}$ . Then*

1. *There exists  $f \in C^\infty(\mathbb{R}^n)$  such that  $Z_0 = \{x \in \mathbb{R}^n : f(x) > 0\}$ , and  $\text{bdy}(Z_0) = \{x \in \mathbb{R}^n : f(x) = 0\}$ .*

2. *There exists  $u_0 \in \mathcal{B}(f|_{\Gamma})$  such that*

$$\int_{\Omega} |Du_0| \, d\mathcal{L}^n = \inf \left\{ \int_{\Omega} |Du| \, d\mathcal{L}^n : u \in \mathcal{B}(f|_{\Omega}) \right\}.$$

The other result that will be of importance to us is from Bombieri, De Giorgi, and Giusti [BDGG69], giving that the fibers of functions of least gradient are least area surfaces.

**Theorem 2.3** ([BDGG69]). *Let  $f$  be a function of least gradient with respect to an open set  $\Omega \subset \mathbb{R}^n$  and let*

$$E_{\lambda} = \{x \in \Omega : f(x) \geq \lambda\}.$$

*Then the set  $E_{\lambda}$  has an oriented boundary of least area with respect to  $\Omega$ .*

Our work in metric spaces will be largely following work by De Pauw and Hardt [DPH12]. However, their work builds off initial work of Ambrosio and Kirchheim [AK00], who constructed the space  $\mathcal{R}_m(X; \mathbb{R})$  of metric currents as (approximately)  $(m+1)$ -linear functionals on the space of  $(m+1)$ -tuples of locally Lipschitz functions. Additional conditions are imposed so that these metric currents exhibit many of the same properties as their Euclidean counterparts.



For a metric space  $X$ , the overarching analogy is roughly that, for a compactly supported  $f \in \text{Lip}(X, \mathbb{R})$  and  $\pi_j \in \text{Lip}_{loc}(X)$ ,  $j = 1, \dots, m$ , the  $(m + 1)$ -tuple  $(f, \pi_1, \dots, \pi_m)$  corresponds to the differential form  $f \, dx_1 \wedge \dots \wedge dx_n$ . Then these metric currents are required to be alternating in the last  $m$  arguments, and vanish if any  $\pi_j$  is constant. Lang [Lan11] later constructed integer rectifiable currents  $\mathcal{R}_m(X; \mathbb{Z})$  in the same manner. De Pauw and Hardt show that their metric currents, which are defined with coefficients in a normed abelian group, are identical to the spaces given by both Ambrosio and Kirchheim, and Lang, provided the appropriate normed abelian group is chosen. More details on the construction of De Pauw and Hardt, which is markedly different from that of the other authors, will be provided in Chapter 6.

# Chapter 3

## Analysis of the Jacobian Integral.

### 3.1 Preliminaries.

We adopt in this section the notation from Jerrard and Soner [JS02], as we must first introduce the notion of a function with bounded  $n$ -variation in order to define our space of admissible functions. We also restate some common theorems, in the context of vector-valued functions.

First, we recall that given  $1 \leq p < \infty$ , positive integers  $m$  and  $k$ , and an open, bounded set  $\Omega \subset \mathbb{R}^m$ , the Sobolev space  $W^{k,p}(\Omega)$  consists of those functions  $f \in L^p(\Omega)$  with the property that the distributional derivative  $D^\alpha f$  exists and is in  $L^p(\Omega; \mathbb{R}^m)$  for each multiindex  $\alpha$  with  $|\alpha| \leq k$ . We further endow  $W^{k,p}(\Omega)$  with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

Now define the class  $W^{k,p}(\Omega; \mathbb{R}^n)$  to be those mappings  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$  whose com-

ponent functions each belong to  $W^{k,p}(\Omega)$ . We equip this space with the norm

$$\|\mathbf{f}\|_{W^{k,p}(\Omega)} := \left( \sum_{j=1}^n \|f^j\|_{W^{k,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

We then note that the Banach-Alaoglu theorem holds for this space – that is, that every bounded set has a weakly convergent subsequence – either by noting that a sequence of functions is bounded in this norm if and only if each of the sequences of component functions is bounded (and hence each of the finitely many components has a weakly convergent subsequence), or by noticing that the space is dual to  $W^{k,q}(\Omega; \mathbb{R}^n)$ , where  $q$  is the arithmetic conjugate of  $p$  (i.e.,  $1/p + 1/q = 1$ ), and applying Banach-Alaoglu directly.

In this section, we denote these (Euclidean) vector-valued functions with a bold typeface (though vectors in the domain and maps into other vector spaces will remain in plain type) and the components with subscripts, as

$$\mathbf{f}(x) = (f_1(x), \dots, f_n(x)).$$

**Definition 3.1** (The function  $j(\mathbf{u})$ ). For sufficiently smooth functions  $\mathbf{u} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , we define the  $n - 1$  form  $j(\mathbf{u})$  and its components  $j^\alpha(\mathbf{u})$  by

$$j(\mathbf{u}) := \sum_{\alpha \in I_{n-1,m}} \det(\mathbf{u}, \mathbf{u}_{x_{\alpha_1}}, \dots, \mathbf{u}_{x_{\alpha_{n-1}}}) dx^\alpha := \sum_{\alpha \in I_{n-1,m}} j^\alpha(\mathbf{u}) dx^\alpha,$$

where  $\alpha$  is a multiindex, and  $I_{n-1,m}$  is the collection of all functions (multiindices) from  $\{1, \dots, n - 1\}$  to  $\{1, \dots, m\}$ .

In the above, “sufficiently smooth” means more precisely, “regular enough that  $j(\mathbf{u}) \in L^1_{loc}(\mathbb{R}^m; \bigwedge^{n-1} \mathbb{R}^n)$ .” We give a short proposition providing justification for our later choices of function spaces:

**Proposition 3.2.** Suppose  $m > n \geq 1$ . If  $\mathbf{u} \in W_{loc}^{1, \frac{mn}{m+1}}(\mathbb{R}^m; \mathbb{R}^n)$ , then  $j(\mathbf{u}) \in L_{loc}^1(\mathbb{R}^m; \bigwedge^{n-1} \mathbb{R}^n)$ .

*Proof.* The proof is a verification of calculations using the Sobolev embedding theorem and Hölder's inequality. Suppose that  $\mathbf{u} \in W_{loc}^{1, \frac{mn}{m+1}}(\mathbb{R}^m; \mathbb{R}^n)$ . Since  $m+1 > n$ , we have

$$\frac{mn}{m+1} < \frac{mn}{n} = m,$$

so by the Sobolev embedding theorem,  $\mathbf{u} \in L_{loc}^{\frac{mn}{m-n+1}}(\mathbb{R}^m; \mathbb{R}^n)$ .

Now,  $j(\mathbf{u})$  is a sum of terms of the form  $\mathbf{u}_{\sigma_1}(\mathbf{u}_{\sigma_2})_{x_{\alpha_1}} \dots (\mathbf{u}_{\sigma_n})_{x_{\alpha_{n-1}}}$ , where  $\sigma \in I_{n,n}$  and  $\alpha \in I_{n-1,m}$ . By the above analysis,  $\mathbf{u}_j \in L_{loc}^{\frac{mn}{m-n+1}}(\mathbb{R}^m)$  for all  $j = 1, \dots, n$ , and  $(\mathbf{u}_j)_{x_k} \in L_{loc}^{\frac{mn}{m+1}}(\mathbb{R}^m)$  for all  $j = 1, \dots, n$ ,  $k = 1, \dots, m$ . Then by Hölder's inequality, the coefficients of the terms in  $j(\mathbf{u})$  are in  $L_{loc}^1(\mathbb{R}^m)$ , since

$$\frac{m-n+1}{mn} + (n-1)\frac{m+1}{mn} = 1.$$

The result follows. □

From here we define the distributional Jacobian:

**Definition 3.3** (Distributional Jacobian). When  $j(\mathbf{u}) \in L_{loc}^1(\mathbb{R}^m; \bigwedge^{n-1} \mathbb{R}^n)$ , define

$$[J\mathbf{u}] := \frac{1}{n} dj(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^m \sum_{\alpha \in I_{n-1,m}} \partial_{x_i} j^\alpha(\mathbf{u}) dx^i \wedge dx^\alpha,$$

in the sense of distributions. That is, for any  $\omega \in C_c^1(\mathbb{R}^m; \bigwedge^n \mathbb{R}^n)$ ,

$$\langle \omega, [J\mathbf{u}] \rangle = \frac{1}{n} \int d^* \omega \cdot j(\mathbf{u}),$$

where  $d^*$  is the formal adjoint of  $d$ .

Intuitively, one thing of  $[J\mathbf{u}]$  as the pull-back of the volume form on  $\mathbb{R}^n$  via the function  $\mathbf{u}$ . Again we will define  $[J^\alpha \mathbf{u}]$ , so that we may write  $[J\mathbf{u}]$  componentwise as

$$[J\mathbf{u}] = \sum_{\alpha \in I_{n,m}} [J^\alpha \mathbf{u}] dx^\alpha.$$

Finally, we define the space of functions of locally bounded  $n$ -variation:

**Definition 3.4.** Let  $\Omega$  be an open subset of  $\mathbb{R}^m$ . We say that a function  $\mathbf{u}$  has locally bounded  $n$ -variation in  $\Omega$  if for every bounded open set  $V \subset \Omega$  there exists some constant  $C = C(V)$  such that

$$\langle \omega, [J\mathbf{u}] \rangle \leq C \|\omega\|_{C^0(\Omega; \bigwedge^n \mathbb{R}^n)}$$

for any  $\omega \in C_c^1(V; \bigwedge^n \mathbb{R}^m)$ . When this holds we write  $\mathbf{u} \in BnV_{loc}(\Omega; \mathbb{R}^n)$ .

The first thing to point out is that  $BnV_{loc}(\Omega; \mathbb{R}^n)$  is not a linear space, which is intuitively reasonable, as the requirement to be of bounded  $n$ -variation is a nonlinear one. Also notice that if  $\mathbf{u} \in BnV_{loc}(\Omega; \mathbb{R}^n)$ , then the Riesz representation theorem gives a nonnegative Radon measure,  $|J\mathbf{u}|$ , and a  $|J\mathbf{u}|$ -measurable function  $\nu : \Omega \rightarrow \bigwedge^n \mathbb{R}^m$  such that  $|\nu(x)| = 1$  for  $|J\mathbf{u}|$ -almost every  $x$ , and

$$\langle \omega, [J\mathbf{u}] \rangle = \int \omega(x) \cdot \nu(x) d|J\mathbf{u}|,$$

for each  $\omega \in C_c^1(V; \bigwedge^n \mathbb{R}^m)$ .

Then we also have

$$|J\mathbf{u}|(V) = \sup \{ \langle \omega, [J\mathbf{u}] \rangle \mid \omega \in C_c^1(V; \bigwedge^n \mathbb{R}^m); |\omega| \leq 1 \}. \quad (3.1)$$

This is the weakly defined quantity which we seek to minimize in this section.

We will show the existence of a minimizer using the direct method in the calculus

of variations, for which we need lower semicontinuity of the integrand, as well as some compactness of the function space. For lower semicontinuity, we will use the following result, and then use a space with strong enough convergence properties that the hypotheses are satisfied.

**Lemma 3.5** (Lower semicontinuity, [JS02]). *Suppose  $\mathbf{u}_k \in BnV_{loc}(\mathbb{R}^m; \mathbb{R}^n)$ , and assume that  $\mathbf{u}_k \rightarrow \mathbf{u}$  in  $L^1_{loc}(\mathbb{R}^m; \mathbb{R}^n)$  and  $j(\mathbf{u}_k) \rightharpoonup j(\mathbf{u})$  weakly in  $L^1_{loc}(\mathbb{R}^m; \mathbb{R}^n)$ .*

*Then  $\mathbf{u} \in BnV_{loc}(\mathbb{R}^m; \mathbb{R}^n)$ , and*

$$|J\mathbf{u}|(V) \leq \liminf_k |J\mathbf{u}_k|(V)$$

*for every open set  $V \subset \mathbb{R}^m$ .*

This result follows from taking  $\omega \in C^1_C(V; \bigwedge^n \mathbb{R}^m)$ ,  $|\omega| \leq 1$ , and the calculation

$$\begin{aligned} |J\mathbf{u}|(V) &\leq \frac{1}{n} \int_V d^* \omega \cdot j(\mathbf{u}) \quad (\text{by 3.1}) \\ &= \lim_{k \rightarrow \infty} \frac{1}{n} \int_V d^* \omega \cdot j(\mathbf{u}_k) \quad (\text{by weak convergence}) \\ &= \lim_{k \rightarrow \infty} \langle \omega, [J\mathbf{u}_k] \rangle \quad (\text{by definition}) \\ &\leq \liminf_{k \rightarrow \infty} |J\mathbf{u}_k|(V) \quad (\text{by 3.1 again}). \end{aligned}$$

We also make use of the following chain rule and weak version of the coarea formula, which in turn use the following notation:

$$\mathbf{u}_a(x) := \frac{\mathbf{u}(x) - a}{|\mathbf{u}(x) - a|} \text{ for } \mathbf{u} : \mathbb{R}^m \rightarrow \mathbb{R}^n, \ a \in \mathbb{R}^n.$$

**Theorem 3.6** (Chain Rule, coarea formula, [JS02]). *If  $\mathbf{u} \in W^{1,n-1}_{loc} \cap L^\infty_{loc}(\Omega; \mathbb{R}^n)$ , then  $\mathbf{u}_a \in W^{1,n-1} \cap L^\infty(\Omega; \mathbb{R}^n)$  for almost every  $a \in \mathbb{R}^n$ , and*

$$[J\mathbf{u}] = \frac{1}{\alpha(n)} \int_{\mathbb{R}^n} [J\mathbf{u}_a] \, da$$

in the sense of distributions (with  $\alpha(n)$  we denote the Lebesgue measure of the unit ball in  $\mathbb{R}^n$ ). If  $\mathbf{u} \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  for  $p > n - 1$  and  $F \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$ , then

$$\langle [J(F \circ \mathbf{u})], \omega \rangle = \frac{1}{\alpha(n)} \int_{\mathbb{R}^n} |JF(a)| \langle [J\mathbf{u}_a], \omega \rangle da, \quad (3.2)$$

for all  $\omega \in C_c^1(\mathbb{R}^m; \bigwedge^n \mathbb{R}^m)$ .

If, in addition,  $\mathbf{u} \in BnV(\mathbb{R}^m; \mathbb{R}^n)$  and either  $\mathbf{u} \in W^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$  for  $p > n$  or  $|\mathbf{u}| = 1$  a.e., then  $\mathbf{u}_a \in BnV(\mathbb{R}^m; \mathbb{S}^{n-1})$  for a.e.  $a \in \mathbb{R}^n$ , and

$$|J\mathbf{u}|(V) = \frac{1}{\alpha(n)} \int_{\mathbb{R}^n} |J\mathbf{u}_a|(V) da \quad (3.3)$$

for every Borel set  $V \subset \mathbb{R}^m$ .

The chain rule above allows us to deduce a uniform bound on the  $L^\infty$  norm of candidate minimizers in  $W^{1,p} \cap L^\infty$  for all  $p$ :

**Corollary 3.7** ( $L^\infty$  bound on candidates.). *Suppose  $\Omega \subset \mathbb{R}^m$  is open and bounded,  $\mathbf{g} \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  for  $p > n - 1$ , and  $\mathbf{f} \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  with  $\mathbf{f}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$ . Then if  $\|\mathbf{g}\|_{L^\infty(\Omega)} = C$ , there exists an  $\tilde{\mathbf{f}} \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  with*

1.  $\|\tilde{\mathbf{f}}\|_{L^\infty(\Omega)} \leq C$ ,
2.  $\tilde{\mathbf{f}}(x) \equiv \mathbf{f}(x)$  for all  $x \in B(0, C)$ ,
3.  $\tilde{\mathbf{f}}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$ , and
4.  $|J\tilde{\mathbf{f}}|(\Omega) \leq |J\mathbf{f}|(\Omega)$ .

*Proof.* Our strategy will be to compose the function  $\mathbf{f}$  with a radial projection, then

to use the chain rule 3.6 to compute  $|J\tilde{\mathbf{f}}|(\Omega)$ . Specifically, define  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$F(x) = \begin{cases} C \frac{x}{|x|} & \text{for } |x| > C \\ x & \text{for } |x| \leq C \end{cases},$$

then define  $\tilde{\mathbf{f}} := F \circ \mathbf{f}$ . Since this mapping contracts all of  $\mathbb{R}^n$  onto the ball of radius  $C$  while leaving the interior fixed, assertions 1-3 hold immediately (note in particular that the image of the boundary of  $\Omega$  lies in the ball of radius  $C$ , since  $\mathbf{g}$  is essentially bounded by  $C$ ). To prove the final assertion, we compute

$$\frac{\partial F_j}{\partial x_k} = \frac{\delta_{j,k}|x|^2 - x_k x_j}{|x|^3} \text{ for all } |x| > C.$$

Then for  $|x| > C$ ,

$$\begin{aligned} |JF(x)| &= \frac{1}{|x|^n} \det \left( I - \frac{x^T x}{|x|^2} \right) \\ &= \frac{1}{|x|^{3n}} (1 - 1) = 0, \end{aligned}$$

using Sylvester's determinant theorem. For  $|x| \leq C$ , the map  $F$  is the identity, and so  $|JF(x)| \equiv 1$ .

Then, applying the chain rule 3.2 we find that for any  $\omega \in C_C^1(\mathbb{R}^m; \bigwedge^n \mathbb{R}^m)$ ,

$$\begin{aligned} \langle [J(F \circ \mathbf{f})], \omega \rangle &= \frac{1}{\alpha(n)} \int_{\mathbb{R}^n} |JF(a)| \langle [J\mathbf{f}_a], \omega \rangle da \\ &= \frac{1}{\alpha(n)} \int_{B(0,C)} \langle [J\mathbf{f}_a], \omega \rangle da \\ &\leq \langle [J\mathbf{f}], \omega \rangle. \end{aligned}$$

Taking the supremum over all such  $\omega$  in the above yields the result.  $\square$

We found no such intrinsic bound on the norm of the distributional derivative.

This uniform bound on the  $L^\infty$ -norm of candidate minimizers is potentially useful in



the search for a minimizer, though the lack of bound on the distributional derivative will lead us to define a class of admissible functions whose Sobolev norm is bounded by hypothesis. Strictly speaking, Corollary 3.7 gives that we need only require that the  $L^p$ -norm of the distributional derivative be uniformly bounded, though then using the boundedness of the domain, one may use a Poincare inequality to come to the same conclusion as Corollary 3.7, without reference to the functions being a minimizing sequence.

## 3.2 Radial minimizers.

We pause here to work through an example of showing that any radial function is a minimizer. Specifically, let  $\mathbf{u} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $n \geq 2$ , be a radial function. That is to say, let

$$\mathbf{u}(x) = \mathbf{u}(r) = (u_1(r), \dots, u_n(r)),$$

where  $r^2 = \sum_{j=1}^m x_j^2$ . For now we will assume that  $\mathbf{u}$  is appropriately smooth, and will remark on the proper space for  $\mathbf{u}$  after the calculations. So

$$\mathbf{u}_{x_j} = \frac{x_j}{r} \mathbf{u}',$$

where

$$\mathbf{u}' := \frac{d\mathbf{u}}{dr} = (u'_1, \dots, u'_n).$$

Hence

$$\begin{aligned} j(\mathbf{u}) &= \sum_{\alpha \in I_{n-1,m}} \det \left( \mathbf{u}, \frac{x_{\alpha_1}}{r} \mathbf{u}', \dots, \frac{x_{\alpha_{n-1}}}{r} \mathbf{u}' \right) dx^\alpha \\ &= \frac{\det(\mathbf{u}, \mathbf{u}', \dots, \mathbf{u}')}{r} \sum_{\alpha \in I_{n-1,m}} \left( \prod_{j=1}^{n-1} x_{\alpha_j} dx^\alpha \right). \end{aligned}$$

Then it follows that

$$j(\mathbf{u}) = \begin{cases} \frac{\det(\mathbf{u}, \mathbf{u}')}{r} \sum_{j=1}^m x_j dx^j & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases}$$

Hence, for  $n > 2$ , any radial map  $\mathbf{u} : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a coarea minimizer, since the distributional Jacobian will be 0. Note in this case, we only need  $\mathbf{u} : \mathbb{R} \rightarrow \mathbb{R}^n$  to have a weak derivative, so that  $\mathbf{u}'$  is defined, since that determinant will be everywhere zero.

Now when  $n = 2$ , we may be more explicit, and write  $\mathbf{u}(r) = (u_1(r), u_2(r))$ , so that

$$\det(\mathbf{u}, \mathbf{u}') = u_1 u_2' - u_2 u_1',$$

and

$$j(\mathbf{u}) = \frac{(u_1 u_2' - u_2 u_1')}{r} (x_1 dx^1 + \dots + x_m dx^m).$$

In this case, we require that  $u_i(r)u_j(r)' \in L^1(\mathbb{R}; \mathbb{R}^m)$ . This is satisfied in particular when  $u_j \in W^{1,p}(\mathbb{R}; \mathbb{R}^m)$  for all  $j$  and any  $p > 1$ , by the Sobolev embedding theorem, which gives in particular that

$$W^{1,p}(\mathbb{R}; \mathbb{R}^m) \hookrightarrow C_B^1(\mathbb{R}),$$

when  $p > 1$ . We conclude that

$$[Ju] = \frac{u_1 u'_2 - u_2 u'_1}{2} \sum_{j,k=1}^m (\partial_{x_j} x_k) dx^j \wedge dx^k = 0.$$

Hence we have shown:

**Proposition 3.8.** *Suppose  $m > n \geq 2$  and  $p > 1$ . Then any radial function  $\mathbf{u} : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  with  $\mathbf{u} \in W^{1,p}(\mathbb{R}^m; \mathbb{R}^n)$  has Jacobian integral equal to zero. In particular, such functions minimize the Jacobian integral.*

### 3.3 Existence of a minimizer for maps from $\mathbb{R}^m \rightarrow \mathbb{R}^2$ .

As mentioned before, one reason the study of the Jacobian integral is interesting is that it has both an analytic interpretation using the integral of the Jacobian, and a geometric one, using the measure of the fibers of the map. Here we will use analytic approach to prove existence of minimizers.

We will minimize over the set  $BnV(\Omega; \mathbb{R}^n)$  from [JS02],[DL03] of functions with distributional Jacobians. Specifically, for  $\Omega \subset \mathbb{R}^m$ , define

$$\mathcal{F}_C^{m,n}(\Omega, \mathbb{R}^n) = \{\mathbf{f} \in W^{1, \frac{mn}{m+1}} \cap BnV(\Omega; \mathbb{R}^n) : \|\mathbf{f}\|_{W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)} < C\}.$$

Notice that we are careful to only discuss minimizers of the Jacobian integral (or, more precisely, the weakly defined analogue). In particular, the coarea formula does not hold in general in  $W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)$  as there are examples [MSZ02] of continuous maps  $\mathbf{f} \in W_{loc}^{1,m}(\mathbb{R}^n; \mathbb{R}^m)$  with almost everywhere vanishing Jacobians and that maps each set  $[0, 1] \times \mathbb{R}^{n-m}$  onto an  $m$ -cube, which would contradict the coarea formula.

**Remark 3.9.** *We use, in general, Dirichlet boundary data, and will require that the boundary be nonempty. For example, one may consider the problem of minimizing the integral of the Jacobian of a map  $\phi : (M^m, g) \rightarrow (N^n, h)$  between compact Riemannian manifolds. Note then that for any real number  $\lambda \neq 0$ ,*

$$\int_M \sqrt{\det(D(\lambda\phi)D(\lambda\phi)^T)} = |\lambda|^n \int_M \sqrt{\det(D\phi D\phi^T)}.$$

*Hence, letting  $\lambda \rightarrow 0$ , we find that  $\phi$  is a coarea minimizer if and only if the energy of  $\phi$  is zero. Due to this observation, we will not hesitate to require that the boundary of  $\Omega$  is nonempty. We deal with this nuance in section 4.3. In particular, we verify that the Hopf fibration  $p : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  is locally a coarea minimizer, in that restricting  $p$  to an open subset with nonempty boundary, we have a coarea minimizer.*

**Theorem 3.10** (Existence for 2-dimensional range.). *Let  $\Omega$  be an open and bounded subset of  $\mathbb{R}^m$  with Lipschitz boundary,  $C > 0$  be constant, and  $\mathbf{g} \in \mathcal{F}_C^{m,2}(\Omega, \mathbb{R}^2)$  be given. Then there exists a function  $\mathbf{u} \in \mathcal{F}_C^{m,2}(\Omega, \mathbb{R}^2)$  with  $\mathbf{u}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$  so that  $|J\mathbf{u}|(\Omega) \leq |J\mathbf{w}|(\Omega)$  for all  $\mathbf{w} \in \mathcal{F}_C^{m,2}(\Omega, \mathbb{R}^2)$  with  $\mathbf{w}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$ .*

**Remark 3.11.** *In this proof, we will proceed for general maps  $\mathbf{f} : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ , rather than substituting  $n = 2$ , as this will make the difficulty of extending this result to higher dimensional ranges more transparent. We use a number of embedding theorems, and with the function spaces used, the last calculation will require that  $(m+1)(n-2) = 0$ , from which we conclude that  $n = 2$ . In Chapter 5 we demonstrate an argument for existence using currents where the space of functions is smaller, but extends to any dimension in the range.*

*Proof.* Let

$$M := \inf\{|J\mathbf{w}|(\Omega) : \mathbf{w} \in \mathcal{F}_C^{m,n}(\Omega, \mathbb{R}^n), \mathbf{w}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}\}.$$

Now let  $\{\mathbf{u}^k\}_{k=1}^\infty \subset \mathcal{F}_C^{m,n}(\Omega, \mathbb{R}^n)$  be a sequence of functions so that

$$\lim_{k \rightarrow \infty} |J\mathbf{u}^k|(\Omega) \rightarrow M.$$

Since

$$\sup_k \|\mathbf{u}^k\|_{W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)} < C,$$

Banach-Alaoglu gives a subsequence, not relabeled, so that  $\mathbf{u}^k \rightharpoonup \mathbf{u}$  weakly in  $W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)$ .

By trace theory [AF03], we will also have that  $\mathbf{u}|_{\partial\Omega} = \mathbf{g}|_{\partial\Omega}$ . We need still show that  $\mathbf{u} \in BnV(\Omega; \mathbb{R}^n)$  so that  $\mathbf{u}$  is in the set of admissible functions, and to show that the distributional Jacobian is lower semicontinuous with respect to this sequence, so that  $|J\mathbf{u}| = M$ . Theorem 3.5 will provide both these results, but we need to show that the hypotheses hold, i.e., that  $\mathbf{u}^k \rightarrow \mathbf{u}$  in  $L^1(\Omega; \mathbb{R}^n)$ , and  $j(\mathbf{u}^k) \rightharpoonup j(\mathbf{u})$  weakly in  $L^1(\Omega)$ .

Now by Rellich-Kondrachov,  $W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)$  embeds compactly into  $L^q(\Omega; \mathbb{R}^n)$ , where  $q < \frac{mn}{m-n+1}$ . In particular, notice that  $1 < \frac{mn}{m-n+1}$ , since  $n \geq 2$  so that  $0 < m \cdot 1 \leq (m+1)(n-1) = mn - (m-n+1)$ . Also using the boundedness of  $\Omega$ , we have  $W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n) \subset\subset L^1(\Omega; \mathbb{R}^n)$ , and may extract another subsequence, again not relabeled, so that

$$\mathbf{u}^k \rightarrow \mathbf{u} \text{ strongly in } L^1(\Omega; \mathbb{R}^n).$$

We next show that  $j(\mathbf{u}^k) \rightharpoonup j(\mathbf{u})$  in  $L^1(\Omega)$ . Writing  $\mathbf{u}^k = (u_1^k, \dots, u_n^k)$  and ex-

panding the determinant, we have that

$$j(\mathbf{u}) = \sum_{\alpha \in I_{n-1,m}} \left( \sum_{\sigma \in I_{n,n}} \text{sgn}(\sigma) (u_{\sigma_1})(u_{\sigma_2})_{x_{\alpha_1}} \cdots (u_{\sigma_n})_{x_{\alpha_{n-1}}} \right) dx^\alpha, \quad (3.4)$$

where  $I_{n,m}$  denotes the collection of all functions from  $\{1, \dots, n\} \rightarrow \{1, \dots, m\}$ . Thus it suffices to show that

$$(u_{\sigma_1}^k)(u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} \rightharpoonup (u_{\sigma_1})(u_{\sigma_2})_{x_{\alpha_1}} \cdots (u_{\sigma_n})_{x_{\alpha_{n-1}}}$$

weakly in  $L^1(\Omega)$ . Expanding as usual and letting  $\phi \in L^\infty(\Omega)$ , we write

$$\begin{aligned} & \int_{\Omega} ((u_{\sigma_1}^k)(u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} - (u_{\sigma_1})(u_{\sigma_2})_{x_{\alpha_1}} \cdots (u_{\sigma_n})_{x_{\alpha_{n-1}}}) \phi \, dx \\ &= \int_{\Omega} ((u_{\sigma_1}^k)(u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} - (u_{\sigma_1})(u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}}) \phi \, dx \\ &+ \int_{\Omega} ((u_{\sigma_1})(u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} - (u_{\sigma_1})(u_{\sigma_2})_{x_{\alpha_1}} \cdots (u_{\sigma_n})_{x_{\alpha_{n-1}}}) \phi \, dx \\ &:= I_k + J_k. \end{aligned}$$

Now we will show that  $\lim_{k \rightarrow \infty} I_k = 0$ , and  $\lim_{k \rightarrow \infty} J_k = 0$ .

**Convergence of the  $I_k$ 's.** To check convergence of the  $I_k$ 's, notice that the product

$(u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}}$  is in  $L^{\frac{mn}{(n-1)(m+1)}}(\Omega)$ , since each member of the product is in  $L^{\frac{mn}{m+1}}(\Omega)$ . Thus Hölder's inequality gives

$$I_k \leq \left( \int_{\Omega} |(u_{\sigma_1}^k - u_{\sigma_1}) \phi|^{\frac{mn}{m-n+1}} \, dx \right)^{\frac{m-n+1}{mn}} \| (u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} \|_{L^{\frac{mn}{(n-1)(m+1)}}(\Omega)}.$$

First we must show that the  $L^{\frac{mn}{(n-1)(m+1)}}$  norm of the product of the derivatives is uniformly bounded. This estimate follows from the condition that  $\mathbf{u}^k \in$

$\mathcal{F}_C^{m,n}(\Omega, \mathbb{R}^n)$ , so  $\|\mathbf{u}^k\|_{W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)} < C$  for all  $k$ . Specifically, Hölder's gives

$$\begin{aligned} \left\| \prod_{j=1}^{n-1} (u_{\sigma_{j+1}}^k)_{x_{\alpha_j}} \right\|_{L^{\frac{mn}{(n-1)(m+1)}}(\Omega)} &\leq \prod_{j=1}^{n-1} \left\| (u_{\sigma_{j+1}}^k)_{x_{\alpha_j}} \right\|_{L^{\frac{mn}{m+1}}(\Omega)} \\ &\leq C^{n-1}. \end{aligned}$$

Also note that

$$\left( \int_{\Omega} |(u_{\sigma_1}^k - u_{\sigma_1})\phi|^{\frac{mn}{m-n+1}} dx \right)^{\frac{m-n+1}{mn}} \rightarrow 0$$

as  $k \rightarrow \infty$ , since, by Rellich-Kondrachov,  $W^{1, \frac{mn}{m+1}}(\Omega)$  embeds continuously into  $L^{\frac{mn}{m-n+1}}(\Omega)$ , so

$$\lim_{k \rightarrow \infty} \|(u_{\sigma_1}^k - u_{\sigma_1})\phi\|_{L^{\frac{mn}{m-n+1}}(\Omega)} \leq C \lim_{k \rightarrow \infty} \|(u_{\sigma_1}^k - u_{\sigma_1})\phi\|_{W^{1, \frac{mn}{m+1}}(\Omega)} = 0.$$

Hence  $I_k \rightarrow 0$  as  $k \rightarrow \infty$ .

**Convergence of the  $J_k$ 's.** Recall that

$$J_k = \int_{\Omega} ((u_{\sigma_2}^k)_{x_{\alpha_1}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} - (u_{\sigma_2})_{x_{\alpha_1}} \cdots (u_{\sigma_n})_{x_{\alpha_{n-1}}}) u_{\sigma_1} \phi \, dx.$$

We expand  $J_k$  again, as

$$\begin{aligned} J_k &= \int_{\Omega} ((u_{\sigma_2}^k)_{x_{\alpha_1}} - (u_{\sigma_2})_{x_{\alpha_1}}) u_{\sigma_1} \phi \left( (u_{\sigma_3}^k)_{x_{\alpha_2}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} \right) dx \\ &\quad + \int_{\Omega} ((u_{\sigma_3}^k)_{x_{\alpha_2}} \cdots (u_{\sigma_n}^k)_{x_{\alpha_{n-1}}} - (u_{\sigma_3})_{x_{\alpha_2}} \cdots (u_{\sigma_n})_{x_{\alpha_{n-1}}}) (u_{\sigma_1} (u_{\sigma_2})_{x_{\alpha_1}} \phi) dx \\ &= I_k^2 + J_k^2. \end{aligned}$$

Indeed, we may continue to expand so that

$$J_k = I_k^2 + I_k^3 + \cdots + I_k^n,$$

where

$$I_k^j := \int_{\Omega} ((u_{\sigma_j}^k)_{x_{\alpha_{j-1}}} - (u_{\sigma_j})_{x_{\alpha_{j-1}}}) u_{\sigma_1} \phi \left( \prod_{\ell=2}^{j-1} (u_{\sigma_\ell})_{x_{\alpha_{\ell-1}}} \right) \left( \prod_{\ell=j+1}^n (u_{\sigma_\ell}^k)_{x_{\alpha_{\ell-1}}} \right) dx.$$

Hence, it suffices to show that  $I_k^j \rightarrow 0$  as  $k \rightarrow \infty$  for each  $j = 2, \dots, n$ . Again, we will apply Hölder's inequality, finding that

$$\begin{aligned} I_k^j &\leq \|((u_{\sigma_j}^k)_{x_{\alpha_{j-1}}} - (u_{\sigma_j})_{x_{\alpha_{j-1}}}) (u_{\sigma_1} \phi)\|_{L^{\frac{mn}{2m-n+2}}(\Omega)} \\ &\quad \cdot \left\| \prod_{\ell=2}^{j-1} (u_{\sigma_\ell})_{x_{\alpha_{\ell-1}}} \prod_{\ell=j+1}^n (u_{\sigma_\ell}^k)_{x_{\alpha_{\ell-1}}} \right\|_{L^{\frac{mn}{(n-2)(m+1)}}(\Omega)}. \end{aligned} \quad (3.5)$$

Further, we have the estimate

$$\begin{aligned} &\left\| \prod_{\ell=2}^{j-1} (u_{\sigma_\ell})_{x_{\alpha_{\ell-1}}} \prod_{\ell=j+1}^n (u_{\sigma_\ell}^k)_{x_{\alpha_{\ell-1}}} \right\|_{L^{\frac{mn}{(n-2)(m+1)}}(\Omega)} \\ &\leq \prod_{\ell=2}^{j-1} \left\| (u_{\sigma_\ell})_{x_{\alpha_{\ell-1}}} \right\|_{L^{\frac{mn}{m+1}}(\Omega)} \prod_{\ell=j+1}^n \left\| (u_{\sigma_\ell}^k)_{x_{\alpha_{\ell-1}}} \right\|_{L^{\frac{mn}{m+1}}(\Omega)} \\ &\leq C^{m-2}, \end{aligned}$$

as may be seen by Hölder's inequality, and our hypothesis of a  $W^{\frac{mn}{m+1}}$  bound on the norm of  $\mathbf{u}$ . Hence, the right term in the product of Equation 3.5 is uniformly bounded.

Now it remains to show that the left term of the product goes to zero. But we have that the product

$$u_{\sigma_1} \phi \in L^{\frac{mn}{m-n+1}}(\Omega),$$

and that  $(u_{\sigma_j}^k)_{x_{\alpha_{j-1}}} \rightharpoonup (u_{\sigma_j})_{x_{\alpha_{j-1}}}$  as  $k \rightarrow \infty$  weakly in  $L^{\frac{mn}{m+1}}(\Omega)$ . Thus  $u_{\sigma_1} \phi$  serves as the necessary test function for convergence only if

$$\frac{mn}{2m-n+2} = 1,$$



i.e., if  $L^{\frac{mn}{2m-n+2}}(\Omega) = L^1(\Omega)$ . This occurs precisely when  $n = 2$  (there is also a solution when  $m = -1$ , but the dimension must be positive).

Hence Theorem 3.5 holds, and we conclude that

$$M \leq |J\mathbf{u}|(\Omega) \leq \lim_{k \rightarrow \infty} |J\mathbf{u}_k| = M,$$

and that  $\mathbf{u}$  is a minimizer. □

In the above theorem, we did not need the full hypothesis  $\|\mathbf{u}\|_{W^{1, \frac{mn}{m+1}}(\Omega; \mathbb{R}^n)} \leq C$ , it was only required that  $\|Du_j\|_{L^{\frac{mn}{m+1}}(\Omega; \mathbb{R}^m)} \leq C$  for each  $j = 1, \dots, n$ , and that the given candidate function was bounded (in  $L^p$ ,  $p > 1$ , and  $L^\infty$ ), as evidenced by Corollary 3.7. Also, given that we are working in an open, bounded set with Lipschitz boundary, if we only require this weaker hypothesis, one already has – via the Poincare inequality – bounds on the  $L^{\frac{mn}{m+1}}$ -norm of  $\mathbf{u}$  without appealing to this other result.

### 3.3.1 An example of convergence in $W^{1, \frac{2m}{m+1}}$ .

In this section we provide an example where a minimizer exists which is in  $W^{1,p}(\Omega; \mathbb{R}^2)$  for  $p < 2$ , but not  $p \geq 2$ . Specifically, we describe a set  $\Omega \subset \mathbb{R}^2$  and give a sequence of Lipschitz functions  $\mathbf{u}_j : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$  with unbounded  $W^{1,2}$ -norm, but with the property that

$$\lim_{j \rightarrow \infty} |J\mathbf{u}_j| < \lim |J\mathbf{v}|$$

for all  $\mathbf{v} \in W^{1, \frac{3}{2}}(\Omega; \mathbb{R}^2)$ , subject to the same boundary conditions.

First we define  $\Omega \subset \mathbb{R}^3$  using the coordinates  $(x, y, z)$  as a triangular prism whose triangle has base length 2 in the  $x$  direction, height 1 in the  $z$  direction and width

1 in the  $y$  direction, with a smaller prism with base length 1, height  $1/2$  removed from the base. See Figure 3.1 for an illustration. We give the boundary data via the function

$$\mathbf{g}(x, y, z) = (z - |x|, y)$$

We may then define a sequence of functions  $\mathbf{u}_\alpha$  parametrized by  $\alpha \in [0, 1]$  in the following manner: the fibers  $u_\alpha^{-1}(x, y)$  will be the union of two straight lines,  $L_1$  and  $L_2$ :

$$L_1(t) = (1 - t)(x, y, 0) + t \left( 1, y, \frac{2 - \alpha}{2} - (1 - \alpha)x \right),$$

and

$$L_2(t) = (1 - t) \left( 1, y, \frac{2 - \alpha}{2} - (1 - \alpha)x \right) + t(2 - x, 1, 0).$$

In Figure 3.1, the left illustration shows  $\mathbf{u}_{\frac{1}{2}}$ , and the right  $\mathbf{u}_0$ .

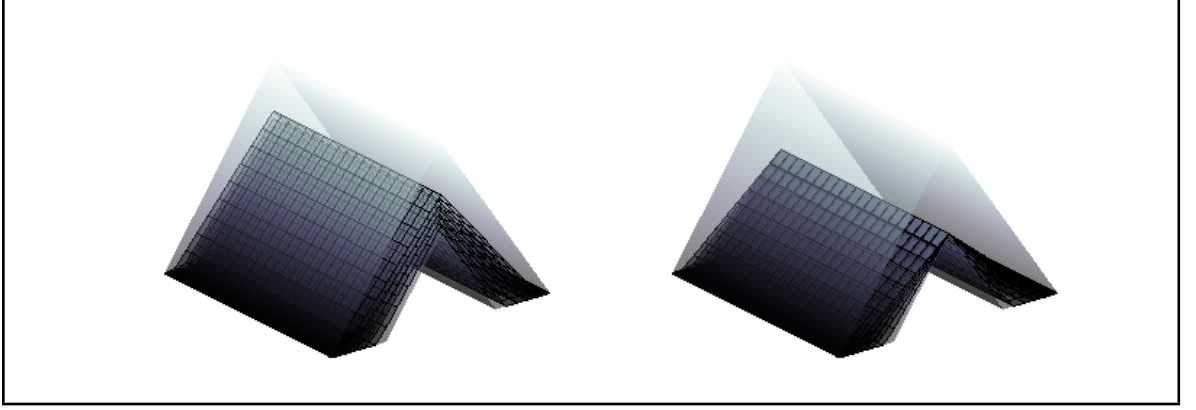
One may then calculate that the sequence  $\mathbf{u}_\alpha$  as  $\alpha \rightarrow 1$  has unbounded  $W^{1,2}$ -norm, though the  $W^{1, \frac{3}{2}}$  norm (indeed the  $W^{1,p}$  norm, for  $p < 2$ ) stays bounded. Roughly, the calculation proceeds by converting the  $y - z$  axis to polar coordinates, and noting that, for fixed  $x$ ,  $|D\mathbf{u}| \approx \frac{1}{r}$ . Then for small  $a$ ,

$$\iiint_0^a |D\mathbf{u}|^p r \, dr d\theta dx \approx \iint r^{1-p} \, dr d\theta dx,$$

an integral which converges whenever  $p < 2$ .

### 3.4 The Euler-Lagrange Equation.

One may calculate that the Euler-Lagrange equations for a function  $\mathbf{u} = (u_1, u_2) : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^2$  are given by



**Figure 3.1:** On left, the fibers of a function converging to the function on the right. Note that the region between the tops of the fibers and the top of the region will all map to the right edge of the rectangle.

$$\begin{cases} \operatorname{div}_x \left( \frac{Du_1 |Du_2|^2 - Du_2}{|J\mathbf{u}|} \right) = 0 \\ \operatorname{div}_x \left( \frac{Du_2 |Du_1|^2 - Du_1}{|J\mathbf{u}|} \right) = 0 \end{cases} \quad (3.6)$$

This is a very nonlinear system of PDE, though we may explicitly compute that affine transformations are stationary points of the functional:

**Example 3.12.** We define an affine function,  $\mathbf{u} : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^2$  by

$$\mathbf{u}(x) = (a_1 \cdot x + b_1, a_2 \cdot x + b_2) = Ax + \mathbf{b}$$

for any  $a_j \in \mathbb{R}^m$ ,  $b_j \in \mathbb{R}$ ,  $j = 1, 2$  (and then taking this as the definition for  $A$ , an  $m \times 2$  matrix, and  $\mathbf{b}$ , a vector in  $\mathbb{R}^2$ ). Then

$$|J\mathbf{u}| \equiv 1,$$

and

$$Du_1 |Du_2|^2 - Du_2 = a_1 |a_2|^2 - a_2,$$

$$Du_2|Du_1|^2 - Du_1 = a_2|a_1|^2 - a_1.$$

The divergence of each of these functions is zero, hence any linear function is a stationary point of the Jacobian integral.

Indeed, any linear function will be a minimizer, since these functions are continuous, so the co-area formula holds. Also, we can compute that the fiber of each point  $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$  will be the set of solutions  $x \in \Omega$  to

$$Ax = \mathbf{y} - \mathbf{b},$$

where  $\mathbf{b} = (b_1, b_2)$  were the constants given earlier. Supposing  $a_1 \neq \lambda a_2$  for any real  $\lambda$ ,  $A$  will have an  $m - 2$  dimensional null space, so the fibers will be linear subspaces. We conclude that the linear function is a coarea minimizer by Theorem 4.7, which makes rigorous the statement that if each fiber of a function  $\mathbf{u}$  is minimal, then  $\mathbf{u}$  is a coarea minimizer.

### 3.5 Gradient flows for nonlinear PDE.

An intriguing approach for linear growth functionals was given by Hardt and Zhou in [HZ94], which studies the gradient flow

$$\frac{\partial u}{\partial t} = \operatorname{div}_x \phi_p(\nabla u),$$

where  $\phi$  is a convex function,  $\phi(0) = 0$ , satisfying a linear growth condition. The Euler-Lagrange equation 3.6 satisfies all these conditions, except in being identically zero in singular directions.

Hardt and Zhou in fact show that, for time independent boundary data, the solutions  $u_t$  converge to a function which is constant in time. That is, which satisfies  $\operatorname{div}_x \phi_p(\nabla u) = 0$ , and is thus a critical point for the integrand  $\int \phi(\nabla u)$ . This makes for another interesting question to pursue:

**Question 3.13.** *Is there a way to solve the Euler-Lagrange equations for the Jacobian integral using a gradient flow?*

A positive answer to this question would give a computational approach to finding minimizers.

### 3.6 Non-uniqueness.

Given the nonlinearity of the integrand, it should not be surprising that minimizers are not in general unique. We will describe an example of a family of coarea minimizers demonstrating this phenomena. The strategy will be to define a coarea minimizer  $\mathbf{f} : C \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , then note that we may change  $f$  on a set of positive 3-dimensional measure without changing  $f^{-1}$  on a set of positive 2-dimensional measure.

First we define a general map  $F_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , by

$$F_\theta(x, y, z) = \frac{(x + z(x(\cos \theta - 1) - y \sin \theta), y + z(y(\cos \theta - 1) + x \sin \theta))}{2(\cos \theta - 1)(z - z^2) + 1}.$$

This map has the property that

$$F_\theta^{-1}(u, v) = \{t(u \cos \theta + v \sin \theta, -u \sin \theta + v \cos \theta, 1) + (1 - t)(u, v, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}.$$

That is to say, the inverse image of a point  $(u, v)$  is a line passing through  $(u, v, 0)$

and  $(u \cos \theta + v \sin \theta, -u \sin \theta + v \cos \theta, 1) = R_\theta \cdot (u, v, 1)^T$ , where

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the matrix which rotates the  $x$ - $y$  plane by  $\theta$  in  $\mathbb{R}^3$ .

Now to define our minimizers, let  $C \subset \mathbb{R}^3$  be the solid unit cylinder,  $B^2(0, 1) \times [0, 1]$ , fix a  $\theta \in [0, 2\pi)$ , and let our boundary data be given by the map  $\mathbf{f}_\theta : C \rightarrow B(0, 1) \subset \mathbb{R}^2$  defined by restricting  $F_\theta$ :

$$\mathbf{f}_\theta(x, y, z) = \begin{cases} F_\theta(x, y, z) & \text{if } |F_\theta(x, y, z)| \leq 1 \\ F_\theta(cx, cy, z) & \text{if } |F_\theta(x, y, z)| > 1 \end{cases},$$

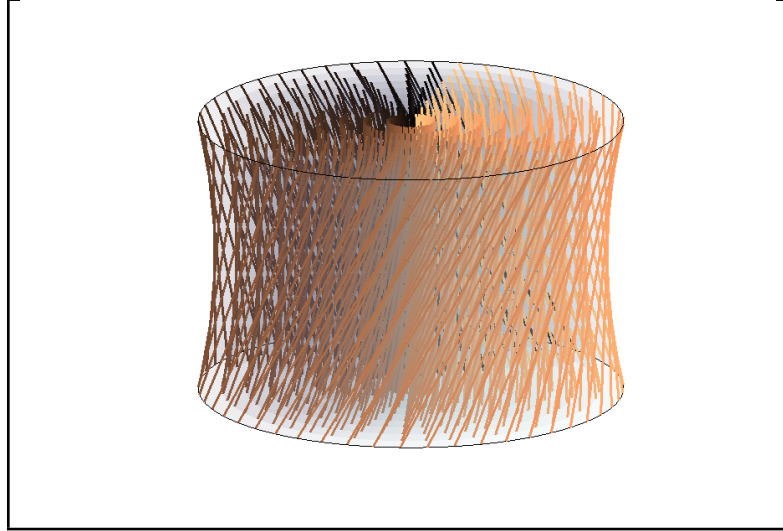
where

$$c := \sqrt{\frac{2(\cos \theta - 1)(z - z^2) + 1}{x^2 + y^2}} = \frac{1}{|F_\theta(x, y, z)|}.$$

Hence, by construction, the inverse image of points in the open unit disk are straight lines – see figure 3.2 – and so  $\mathbf{f}_\theta$  is a coarea minimizer. Also notice that the inverse image of the boundary of the disk are 2-dimensional sets, though since the boundary of the disk has measure zero, this does not effect the coarea. See figure 2.2, in particular the right hand side.

We may view the lack of uniqueness in this problem by recasting it as a problem of Lipschitz extensions. In particular, the set where  $|F_\theta| \leq 1$  and  $0 \leq z \leq 1$  is a subset of a solid hyperboloid of one sheet,  $H$ , which is a subset of the solid unit cylinder  $C$ . Defining

$$\mathbf{h}_\theta(x, y, z) = \mathbf{f}_\theta(x, y, z) \text{ for } (x, y, z) \in H \cup \partial C,$$



**Figure 3.2:** The one dimensional fibers of the minimizing map  $f$  are all straight lines .

then we see that  $\mathbf{h}_\theta^{-1}(u, v)$  is defined for a.e.  $(u, v) \in B(0, 1)$ . More precisely, it is defined everywhere except on the boundary. So every Lipschitz extension of  $\mathbf{h}_\theta$  to all of  $C$  will have the same coarea as  $\mathbf{f}_\theta$ , and so will also be a minimizer of the functional.

# Chapter 4

## Geometry of Coarea Minimizers.

In this chapter, we provide examples of coarea minimizers, as well as methods of producing such functions. This may be interesting in its own right, or we may follow Parks [Par77] who studied total variation minimizers with the goal of computing examples of area minimizers, in light of the work of Bombieri, De Giorgi and Giusti [BDGG69]. The end of the chapter will address the question of the minimality of fibers of coarea minimizers.

### 4.1 Harmonic morphisms, examples of coarea minimizers.

A broad class of examples of coarea minimizers may be generated using *harmonic morphisms*. A harmonic morphism is any map  $\phi : M \rightarrow N$  between Riemannian manifolds so that for each harmonic  $f$  defined on an open  $V \subset N$ , the function  $f \circ \phi$  is harmonic on  $\phi^{-1}(V)$ . The property that makes these maps of interest to us is



Theorem 4.3, roughly that their fibers may be minimal submanifolds. These maps are necessarily smooth, so the coarea formula will hold. A useful characterization of harmonic morphisms is the following:

**Proposition 4.1** (See, for example, [BE81]). *A nonconstant map  $\phi : (M, g) \rightarrow (N, g)$  is a harmonic morphism if and only if  $\phi$  is harmonic and horizontally conformal.*

In the above, *horizontally conformal* means, approximately, that the restrictions to perpendicular slices of fibers are conformal. Explicitly,

**Definition 4.2** (horizontally conformal, horizontally homothetic). Given a map  $\phi : (M, g) \rightarrow (N, h)$ , let  $\mathcal{V}_{\mathbf{x}} := \ker(d\phi_{\mathbf{x}})$ , and  $\mathcal{H}_{\mathbf{x}} := \mathcal{V}_{\mathbf{x}}^{\perp}$ . Then  $\phi$  is horizontally conformal provided  $d\phi_{\mathbf{x}}|_{\mathcal{H}_{\mathbf{x}}} : \mathcal{H}_{\mathbf{x}} \rightarrow N$  is conformal. That is to say, there is a  $\mu : M \rightarrow \mathbb{R}$  so that for each  $v, w \in T_{\mathbf{x}}M$  and  $\mathbf{x} \in M$ ,

$$\mu^2(\mathbf{x})\langle v, w \rangle_{g(\mathbf{x})} = \langle dp_{\mathbf{x}}(v), dp_{\mathbf{x}}(w) \rangle_{h(p(\mathbf{x}))}.$$

In case that  $\phi$  is horizontally conformal and  $\nabla_{\mathcal{H}}(\mu^2) \equiv 0$ , we say that  $\phi$  is *horizontally homothetic*. Here  $\nabla_{\mathcal{H}}(\mu^2)$  is the horizontal projection of  $\nabla\mu^2$  under this orthogonal decomposition, so that

$$\nabla\mu^2 = \nabla_{\mathcal{H}}(\mu^2) + \nabla_{\mathcal{V}}(\mu^2).$$

We also have following theorem that gives conditions for the fibers of a harmonic morphism to be minimal:

**Theorem 4.3.** [BE81] *Let  $\phi : (M, g) \rightarrow (N^n, h)$  be a submersion and a harmonic morphism. Then if  $n = 2$ , the fibers of  $\phi$  are minimal submanifolds. If  $n > 2$ , then the fibers of  $\phi$  are minimal submanifolds if and only if  $\phi$  is horizontally homothetic.*

Hence, for a horizontally homothetic harmonic morphism, we have that

$$\int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f^{-1}(y) \cap \Omega) d\mathcal{L}^n(y),$$

the integral over the fibers, will be minimized.

**Theorem 4.4.** *Suppose there exists a smooth non-constant horizontally homothetic harmonic morphism  $u : \Omega \rightarrow \mathbb{R}^n$  such that  $u|_{\partial\Omega} = g$ . Then  $u$  minimizes the coarea among all smooth functions  $v$  with  $v|_{\partial\Omega} = g$ .*

This may be seen in light of the above Theorem 4.3 and Theorem 4.7. Note that the hypothesis of being horizontally homothetic may be dropped in case that  $n = 2$ .

This result is surprising in light of the following observation: if  $u = (\phi^1, \phi^2) : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^2$  is smooth, then

$$|Ju| = \sqrt{|\nabla\phi^1|^2|\nabla\phi^2|^2 - (\nabla\phi^1 \cdot \nabla\phi^2)^2} \leq |\nabla\phi^1||\nabla\phi^2|.$$

But if  $u$  is horizontally conformal and harmonic (hence a coarea minimizer), then

$$\Delta\phi^1 = \Delta\phi^2 = 0, \quad \langle \nabla\phi^1, \nabla\phi^2 \rangle = 0, \quad \text{and} \quad |\nabla\phi^1|^2 = |\nabla\phi^2|^2.$$

Hence

$$|Ju| = |\nabla\phi^1||\nabla\phi^2| = K^2 \quad (\text{where } K = |\nabla\phi^1| = |\nabla\phi^2|).$$

We conclude that since  $u$  is a coarea minimizer, *any* smooth function  $v = (\psi^1, \psi^2) : \Omega \rightarrow \mathbb{R}^2$  with  $v|_{\partial\Omega} = g$  and  $|\nabla\psi^1||\nabla\psi^2| \leq |\nabla\phi^1||\nabla\phi^2|$  for all  $x \in \Omega$  must in fact have

$$\langle \nabla\psi^1, \nabla\psi^2 \rangle \equiv 0,$$

and

$$|\nabla\psi^1||\nabla\psi^2| \equiv K^2,$$

or else it would contradict  $u$  being a minimizer. This in turn implies that

$$\frac{\partial}{\partial x_j} |\nabla\psi^k| = 0,$$

so  $|\nabla\psi^k|$  is constant in each direction.

Then we also have the following necessary condition on the boundary data  $g$  in order to admit a harmonic morphism as a coarea minimizer:

**Proposition 4.5.** *If  $u : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^2$  is a horizontally homothetic harmonic morphism which minimizes the coarea among all smooth functions  $v$  with  $v|_{\partial\Omega} = g = (g^1, g^2)$ , then  $|\nabla g^1| = |\nabla g^2|$  is constant.*

This follows from the fact that  $|\nabla u|$  is constant on  $\Omega$ , and since the limit on the boundary is a Lipschitz  $g$ , it must in fact be constant.

It is a result of Jacobi (see [BW88], [Jac48]) that any local solution  $z : \mathbb{R}^3 \rightarrow \mathbb{C}$  of

$$\langle f(z(x))[1 - g^2(z(x)), i(1 + g^2(z(x))), 2g(z(x))], x \rangle_{\mathbb{C}} = 1$$

is a harmonic morphism, where  $f, g : \mathbb{C} \rightarrow \mathbb{C}$  are holomorphic. In fact, Baird and Wood [BW88] show that *all* harmonic morphisms from  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$  arise this way (more specifically, morphisms from  $\mathbb{R}^3$  to a 2 dimensional Riemannian manifold). We may use this result to generate coarea minimizers from  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ .

For example, letting  $f_r(z) = -1/(2irz)$ ,  $g(z) = z$  and  $r \in \mathbb{R}_+$ , we find

$$(x_1 - ix_2)z^2 - 2(x_3 + ir)z - (x_1 + ix_2) = 0$$

Hence,

$$z_r(\mathbf{x}) = \frac{x_3 + ir \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2irx_3}}{x_1 - ix_2}$$

is a harmonic morphism, and so a co-area minimizer.

## 4.2 Generating new minimizers.

Other theorems on minimizers of the Jacobian integral are guided by analogous results for TV minimizers (as it is of course necessary that any result for  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  must hold in particular when  $n = 1$ ). One pleasant result is given in [AFP00], that the composition  $v = f \circ \phi$  of a BV function  $f$  with a Lipschitz function  $\phi$  is still of bounded variation, and further, for measurable  $\Omega \subset \mathbb{R}^n$ ,

$$(Lip(\phi))^{1-n} |Df|(\phi(\Omega)) \leq |Dv|(\Omega) \leq (Lip(\phi))^{n-1} \phi |Du|(\phi(\Omega)).$$

One may use the chain rule for distributional Jacobians, Eq. 3.2, to show that when  $n = 2$ , composing minimizers of the Jacobian integral with holomorphic functions creates new minimizers:

**Theorem 4.6.** *Suppose  $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$  has  $|JF(a)| \equiv 1$  for  $a \in \mathbb{R}^n$ ,  $\Omega \subset \mathbb{R}^m$  is open and bounded with Lipschitz boundary, and  $u \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  is a coarea minimizer. Then  $F \circ u : \Omega \rightarrow \mathbb{R}^n$  minimizes  $|Jg|(\Omega)$  among all functions  $g \in W^{1,p} \cap L^\infty(\Omega; \mathbb{R}^n)$  with  $g|_{\partial\Omega} = (F \circ u)|_{\partial\Omega}$ .*

*Proof.* Recall the weak chain rule Eq. 3.2 that says for all 1-forms  $\omega$ ,

$$\begin{aligned} \langle [J(F \circ u)], \omega \rangle &= \frac{1}{\pi} \int_{\mathbb{R}^2} |JF(y)| \langle [Ju_y], \omega \rangle dy \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \langle [Ju_y], \omega \rangle dy = \langle [Ju], \omega \rangle, \end{aligned}$$

and so

$$|J(F \circ u)|(\Omega) = \sup_{\omega} \langle [J(F \circ u)], \omega \rangle = \sup_{\omega} \langle [Ju], \omega \rangle = |Ju|(\Omega).$$

Now suppose for contradiction that there is a  $w \in W^{1,p} \cap L^{\infty}(\Omega; \mathbb{R}^n)$  with the proper trace so that

$$|Jw|(\Omega) < |J(F \circ u)|.$$

Then define  $\hat{w} : \Omega \rightarrow \mathbb{R}^2$  locally by  $\hat{w} = F^{-1} \circ w$ , so  $\hat{w}|_{\partial\Omega} = F^{-1}(F \circ g) = g$ , and the inverse function theorem gives

$$|JF^{-1}| \equiv |JF| \equiv 1.$$

But then

$$\begin{aligned} \sup \langle [Ju], \omega \rangle &= \sup \langle [J(F \circ u)], \omega \rangle \\ &> \langle [Jw], \omega \rangle = \langle [J\hat{w}], \omega \rangle, \end{aligned}$$

a contradiction.

□

### 4.3 The Hopf Fibration.

As an example of the connections between harmonic morphisms and coarea minimizers from Section 4.1, we prove that the Hopf fibration is locally a coarea minimizer (see Remark 3.9). Recall that if we identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$  and  $\mathbb{R}^3$  with  $\mathbb{C} \times \mathbb{R}$ , then the Hopf fibration is the map  $p : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  defined by

$$p(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2),$$

or in Euclidean coordinates by

$$p(x_1, x_2, x_3, x_4) = (2(x_1x_3 + x_2x_4), 2(x_2x_3 - x_1x_4), (x_1^2 + x_2^2 - x_3^2 - x_4^2)).$$

See, for example, [Lyo03] for more detail.

We will calculate explicitly that the Hopf map is a harmonic morphism (and therefore has minimal fibers, since the target space has dimension 2). We do this by computing that  $p$  is horizontally conformal, and then noting that it is harmonic.

We may calculate

$$\frac{1}{2}dp_{\mathbf{x}} = (x_3, -x_4, x_1)dx_1 + (x_4, x_3, x_2)dx_2 + (x_1, x_2, -x_3)dx_3 + (x_2, -x_1, -x_4)dx_4,$$

and so the kernel  $\mathcal{V}_{\mathbf{x}}$  of  $dp_{\mathbf{x}}$  is given by

$$\mathcal{V}_{\mathbf{x}} = \{\lambda(-x_2dx_1 + x_1dx_2 - x_4dx_3 + x_3dx_4) : \lambda \in \mathbb{R}\}.$$

Hence the horizontal component of the tangent space of  $\mathbb{S}^3$ ,  $\mathcal{H}_{\mathbf{x}}$ , is spanned by  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ , where

$$\mathbf{v}_1 = x_3dx_1 + x_4dx_2 + x_1dx_3 + x_2dx_4,$$

$$\mathbf{v}_2 = -x_4dx_1 + x_3dx_2 + x_2dx_3 - x_1dx_4, \text{ and}$$

$$\mathbf{v}_3 = x_1dx_1 + x_2dx_2 - x_3dx_3 - x_4dx_4.$$

In order to check horizontal conformality, we confirm that  $dp_{\mathbf{x}|_{\mathcal{H}_{\mathbf{x}}}} : \mathcal{H}_{\mathbf{x}} \rightarrow T_{p(\mathbf{x})}\mathcal{S}^2$  is surjective and conformal. We confirm this with the calculation

$$\begin{aligned} dp_{\mathbf{x}}(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3) &= a|\mathbf{v}_1|^2dx_1 + b|\mathbf{v}_2|^2dx_2 + c|\mathbf{v}_3|^2 \\ &= adx_1 + bdx_2 + cdx_3, \end{aligned}$$

for all  $a, b, c \in \mathbb{R}$ , since  $\mathbf{x} \in \mathbb{S}^3$ , so  $|\mathbf{v}_j|^2 = 1$  for  $j = 1, 2, 3$ .

Note that the  $\mathbf{v}_j$ 's are orthonormal, so we will have horizontal conformality. Explicitly, we have demonstrated that

$$\begin{aligned} a_1 a_2 dx_1 + b_1 b_2 dx_2 + c_1 c_2 dx_3 &= \langle a_1 \mathbf{v}_1 + b_1 \mathbf{v}_2 + c_1 \mathbf{v}_3, a_2 \mathbf{v}_1 + b_2 \mathbf{v}_2 + c_2 \mathbf{v}_3 \rangle_{\mathbb{R}^4|_{\mathbb{S}^3}} \\ &= \langle dp_{\mathbf{x}}(a_1 \mathbf{v}_1 + b_1 \mathbf{v}_2 + c_1 \mathbf{v}_3), dp_{\mathbf{x}}(a_2 \mathbf{v}_1 + b_2 \mathbf{v}_2 + c_2 \mathbf{v}_3) \rangle_{\mathbb{R}^3|_{\mathbb{S}^2}}. \end{aligned}$$

Hence  $p$  is horizontally conformal and harmonic, and therefore a harmonic morphism. Thus  $p$  is a coarea minimizer. Since  $\mathbb{S}^2$  is a 2-dimensional manifold, Theorem 4.4 applies to any bounded subset of  $\mathbb{S}^3$ , and so  $p$  is a local coarea minimizer.

## 4.4 Fibers of coarea minimizers.

We have used the following theorem a number of times, which we state and prove here for concreteness:

**Theorem 4.7.** *Suppose  $m > n \geq 1$ ,  $\Omega \subset \mathbb{R}^m$  is open and bounded with Lipschitz boundary, and  $f \in C^{m-n+1}(\Omega; \mathbb{R}^n)$  has the property that for  $\mathcal{H}^n$ -almost every  $y \in \mathbb{R}^n$ ,  $f^{-1}(y)$  has least  $(m-n)$ -measure subject to its boundary. Then  $f$  has least coarea among those functions in  $C^{m-n+1}(\Omega; \mathbb{R}^n)$  with the same boundary data.*

*Proof.* Since  $f \in C^{m-n+1}(\Omega; \mathbb{R}^n)$ , Sard's theorem holds, so we know that for almost every  $y \in \mathbb{R}^n$  the Jacobian of the set  $f^{-1}(y)$  has full rank. Then if there is any  $g \in C^{m-n+1}(\Omega; \mathbb{R}^n)$  with less coarea, notice there must be an open set  $U \subset \mathbb{R}^n$  with positive measure so that

$$\int_U \mathcal{H}^{m-n}(\Omega \cap g^{-1}(y)) \, d\mathcal{L}^n(y) < \int_U \mathcal{H}^{m-n}(\Omega \cap f^{-1}(y)) \, d\mathcal{L}^n(y).$$

This is a contradiction, since this implies that the  $(n - m)$ -dimensional measure of the fibers of  $g$  are less than those of  $f$  on a set of non-zero measure.  $\square$

Note that each of the examples in this thesis have this property (though the example from Section 3.3.1 has an obstacle), though many are generated using other methods. For example, Theorem 4.4 is really an alternative characterization of harmonic morphisms whose fibers are minimal.

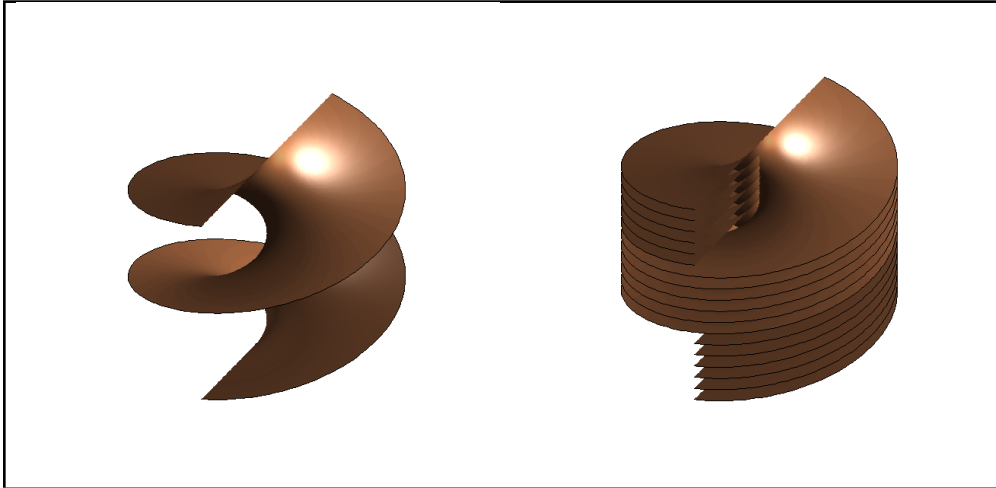
We can also view Theorem 4.7 to as generalizing the example from Section 3.6. Specifically, this example used a ruled (in fact, doubly ruled) surface  $C$ — through each point on  $C$  there is a line which lies entirely in  $C$ . We then used that surface to generate a solid, by varying some parameter (in this case, the radius). Then we define a map on the ruled surface mapping each point from this line to the same place. Hence the fibers of this map (from 3 dimensions down to 2) will each be straight lines — specifically, the rulings of  $C$ .

To generate these examples, recall that a ruled surface  $S$  may be parametrized via

$$S(r, \theta) = p(\theta) + r \cdot c(\theta),$$

where  $0 \leq r, \theta \leq 1$ , (or any other rectangle in  $\mathbb{R}^2$ , via scaling),  $c(\theta)$  is a curve on the sphere giving the direction of the lines of the ruling, and  $p(\theta)$  is a curve on the surface that passes through each of these rulings. One must then find another parameter  $t \in [0, 1]$  to vary so that the surfaces  $S_t \subset \mathbb{R}^3$  are disjoint.





**Figure 4.1:** On the left is a single helicoid, and the right is the family of parametrized helicoids that generate the region  $\Omega$ .

As an example, consider the helicoid, a ruled surface which may be parametrized by

$$H_t(r, \theta) = (0, 0, \theta + t) + r(\cos \theta, \sin \theta, 0),$$

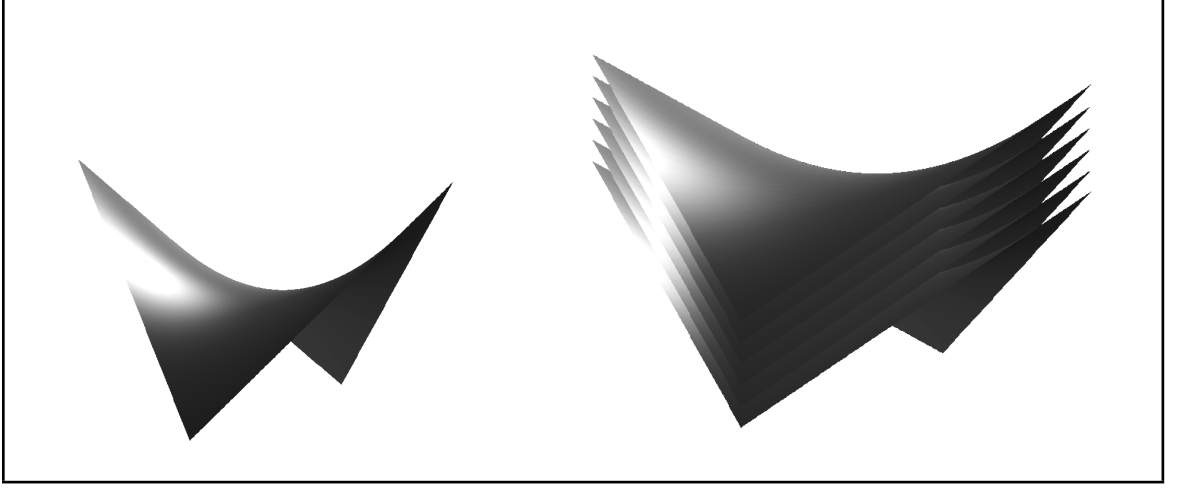
where  $(r, \theta) \in [-1, 1] \times [0, 2\pi]$ , and  $t \in [0, \pi]$ . One may check that the surfaces  $H_t$  are disjoint, see Figure 4.1. Then let

$$\Omega \subset \mathbb{R}^3 := \cup_{t \in [0, \pi]} H_t,$$

and define  $f : \Omega \rightarrow \mathbb{R}^2$  by “projecting” each point back to  $(t, \theta)$ . This map is well-defined, and its fibers are the straight lines of the ruling of the helicoid.

One may also use this method to produce, for example, a domain and a map using a hyperbolic paraboloid (see Figure 4.2), or using a rectangle (viewed as a ruled surface) to generate the example from Chapter 1 (see Figure 1.1).

An especially vexing question is whether the converse of Theorem 4.7 is true. This is a natural question to ask in light of Theorem 2.3 from Bombieri, De Giorgi, and



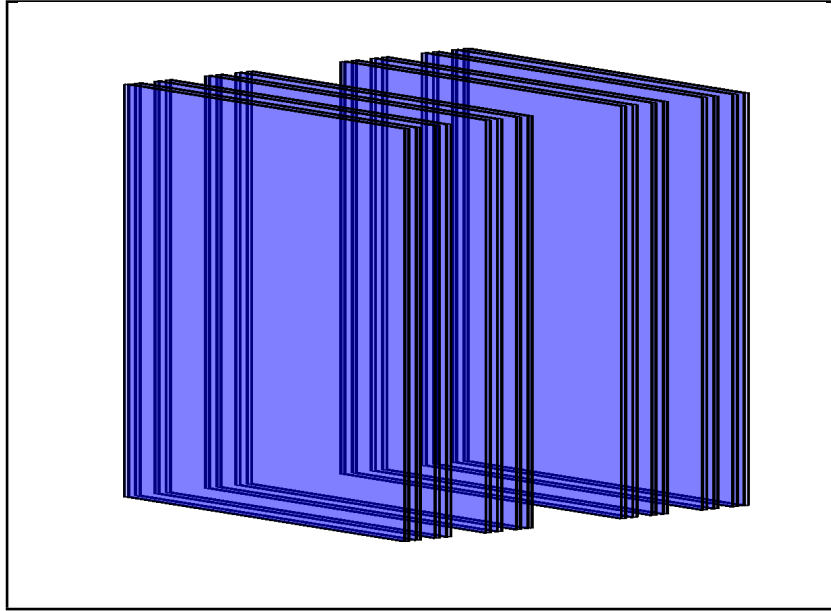
**Figure 4.2:** On the left is a single hyperbolic paraboloid, and the right is the family of parametrized hyperbolic paraboloids that generate a region in space.

Giusti [BDGG69] showing that the fibers of functions of least gradient have least area. At the very least, it seems difficult to find a counterexample. Explicitly, we ask the following question:

**Question 4.8.** *If  $u : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a coarea minimizer, then do almost all of the fibers of  $u$  have least  $\mathcal{H}^{n-m}$  measure subject to the given boundary conditions?*

A positive answer to the above would allow us to infer some regularity of coarea minimizers: by the results of Almgren [AST00], the singular set of each fiber would be of dimension at most  $m - n - 2$ . Note that this would still not imply any regularity for the function itself. This is to be expected, as composing a function with a measure-preserving transformation its range will not change the integral over the fibers of the map (only rearrange the fibers), even if the transformation is very discontinuous.

As a more concrete example, consider the fat Cantor set  $C$ , formed by iteratively removing an interval of length  $2^{-2n}$  from  $[0, 1]$ . Recall that  $\mathcal{L}^1(C) = 1/2$ , but that  $C$



**Figure 4.3:** We may define a very discontinuous function whose fibers are straight lines on the fat Cantor set crossed with a rectangle.

is disconnected (contains no intervals). Hence, we may define a function  $f : C \rightarrow \mathbb{R}$  by

$$f(x) = \mathcal{L}^1(\{y \in C : y \leq x\}).$$

Now let our domain  $\Omega := C \times [0, 1]^2$  be the fat Cantor set crossed with a rectangle, and define a function  $F : [0, 1]^3 \rightarrow \mathbb{R}^2$  by

$$F(x, y, z) = \begin{cases} (f(x), y) & \text{if } (x, y, z) \in \Omega \\ (0, 0) & \text{otherwise.} \end{cases}$$

Now the inverse image of almost every point in  $[0, 1/2] \times [0, 1]$  is a straight line, but  $F$  is discontinuous everywhere away from the origin.

# Chapter 5

## Slices and the Coarea formula.

### 5.1 Flat chains in Euclidean space.

We develop the theory of slices of flat chains in Euclidean space to allow for an approach to the problem using the rectifiable currents of geometric measure theory.

#### 5.1.1 Background on differential forms and currents.

We adopt the standard notation (see, for example, [FX03],[Fed69],[KP08],[Mor09]) and write, for  $U \subset \mathbb{R}^n$  and  $V$  an  $m$ -dimensional vector space,

$$\mathcal{E}(U, V) := C^\infty(U; V).$$

For  $\phi \in \mathcal{E}(U, V)$ , we define the support of  $\phi$  by

$$\text{supp } \phi = U \setminus \bigcup \{W : W \text{ if open, } \phi(x) = 0 \text{ if } x \in W\},$$

so that  $\text{supp } \phi$  is necessarily a closed set. Now write, for each compact  $K \subset U$ ,

$$\mathcal{D}_K(U, V) := \{\phi \in \mathcal{E}(U, V) : \text{supp } \phi \subset K\},$$

and

$$\mathcal{D}(U, V) := \bigcup \{ \mathcal{D}_K(U, V) : K \subset U \text{ is compact} \}.$$

We equip  $\mathcal{E}(U, V)$  with the topology generated by the following family of seminorms. Let  $j \in \mathbb{Z}_+$  and  $K \subset U$  be compact. Then

$$\nu_K^j(\phi) := \sup \{ \|D^k \phi(x)\| : 0 \leq k \leq j, x \in K \}, \quad (5.1)$$

where  $\|D^k \phi(x)\|$  is the operator norm of the  $k$ th differential of  $\phi$  at  $x$ . This family of seminorms induces a locally convex, translation invariant Hausdorff topology on  $\mathcal{E}(U, V)$  with subbasis consisting of sets of the form

$$\{ \phi \in \mathcal{E}(U, V) : \nu_K^j(\phi - \psi) < r \}$$

for a fixed  $\psi \in \mathcal{E}(U, V)$  and  $r > 0$ .

The space  $\mathcal{D}(U, V)$  is given the largest topology such that the inclusion maps  $\iota_K : \mathcal{D}_K(U, V) \hookrightarrow \mathcal{D}(U, V)$  are continuous for each compact  $K \subset U$ .

Now we may define the duals to these spaces. The space  $\mathcal{E}'(U, V)$  is the vector space of continuous real-valued functions on  $\mathcal{E}(U, V)$  and  $\mathcal{D}'(U, V)$  is the vector space of continuous real-valued functions on  $\mathcal{D}(U, V)$ . We equip each of these spaces with the weak-\* topology, so that for  $T_j \in \mathcal{E}'(U, V)$  (resp.  $\mathcal{D}'(U, V)$ )

$$T_j \rightarrow T \text{ if and only if } T_j(\phi) \rightarrow T(\phi)$$

for all  $\phi \in \mathcal{E}(U, V)$  (resp.  $\mathcal{D}(U, V)$ ).

Notice in particular that

$$\mathcal{D}(U, V) \subset \mathcal{E}(U, V), \text{ so } \mathcal{E}'(U, V) \subset \mathcal{D}'(U, V).$$

Finally, we define

$$\mathcal{E}^m(U) := \mathcal{E}(U, \bigwedge^m \mathbb{R}^n), \quad \mathcal{E}_m(U) := \mathcal{E}'(U, \bigwedge^m \mathbb{R}^n),$$

$$\mathcal{D}^m(U) := \mathcal{D}(U, \bigwedge^m \mathbb{R}^n), \quad \mathcal{D}_m(U) := \mathcal{D}'(U, \bigwedge^m \mathbb{R}^n),$$

and will call  $\mathcal{D}^m(U)$  the space of *differential  $m$ -forms*, and  $\mathcal{D}_m(U)$  the space of  *$m$ -dimensional currents*.

For  $T \in \mathcal{D}_m(U)$ , we additionally define the *support* of a current,  $\text{spt } T$  by

$$\text{spt } T := \{x \in U : \forall \epsilon > 0, \exists \phi \in \mathcal{D}^m(U) \text{ with } \text{supp } \phi \subset B(x, \epsilon) \text{ and } T\phi \neq 0\},$$

and the *mass* of a current:

$$\mathbb{M}(T) := \sup\{T(\omega) : \omega \in \mathcal{D}^m(U), \|\phi\| \leq 1\}.$$

The space of *normal currents* are those currents with both finite mass and finite boundary mass (which rules out certain fractals like the Koch snowflake whose boundary has infinite 1-dimensional measure):

$$\mathbb{N}_m(U) = \{T \in \mathcal{D}_m(U) : \mathbb{M}(T) + \mathbb{M}(\partial T) < \infty\}.$$

Finally, we will find use for the space of normal currents supported in a compact set  $K \subset U$ :

$$\mathbb{N}_{m,K}(U) = \mathbb{N}_m(U) \cap \{T \in \mathcal{D}_m(U) : \text{spt } T \subset K\}.$$

### 5.1.2 Basic operations on currents.

The following are some operations on currents we will use to define the slice of a flat chain. We assume the reader is familiar with the (dual) operations on differential forms.

We first define the *interior product* on currents. Specifically, for  $T \in \mathcal{D}_m(U)$ ,  $\phi \in \mathcal{E}^k(U)$  and  $k \leq m$ , we have  $T \lrcorner \phi \in \mathcal{D}_{m-k}(U)$  defined via

$$(T \lrcorner \phi)(\psi) := T(\phi \wedge \psi)$$

for all  $\psi \in \mathcal{D}^{m-k}(U)$ .

Then for  $\xi \in C^\infty(U; \bigwedge_p \mathbb{R}^n)$ , we define the *wedge product* on currents,  $T \wedge \xi \in \mathcal{D}_{m+p}(U)$ , by

$$(T \wedge \xi)(\psi) := T(\xi \lrcorner \psi)$$

for  $\psi \in \mathcal{D}^{m+p}(U)$  (and where  $\xi \lrcorner \psi$  is characterized by  $\langle \xi \lrcorner \psi, \alpha \rangle := \langle \psi, \alpha \wedge \xi \rangle$  for  $\alpha \in \bigwedge_m(\mathbb{R}^n)$ ).

We also define the boundary operator,  $\partial T \in \mathcal{D}_{m-1}(U)$  by

$$(\partial T)(\psi) := T(d\psi),$$

for  $\psi \in \mathcal{D}^{m-1}(U)$  and  $m \geq 1$  (if  $m = 0$ , we may define  $\partial T = 0$ ).

Finally, we define the partial derivative of a current,  $D_{x_j} T \in \mathcal{D}_m(U)$ , coordinate-wise on simple differential forms by, for  $1 \leq k_1 < \cdots < k_m \leq n$ ,

$$D_{x_j} T(\phi \, dx_{k_1} \wedge \cdots \wedge dx_{k_m}) = -T\left(\frac{\partial \phi}{\partial x_j} \, dx_{k_1} \wedge \cdots \wedge dx_{k_m}\right),$$

and extend by linearity to general differential forms.

We also record a list of relations between these operations:

**Proposition 5.1.** *Suppose  $\phi \in \mathcal{E}^k(U)$ ,  $\xi \in C^\infty(U; \bigwedge_p \mathbb{R}^n)$ , and  $T \in \mathcal{D}_m(U)$ . Then*

- if  $m \geq 2$ ,  $\partial(\partial T) = 0$ ;
- $(\partial T) \lrcorner \phi = T \lrcorner d\phi + (-1)^k \partial(T \lrcorner \phi)$ ;

- if  $m \geq 1$ , then  $\partial T = -\sum_{j=1}^n (D_{x_j} T) \lfloor dx_j$ .

### 5.1.3 Rectifiable sets.

Our next aim is to define a current associated with manifold-like subsets of Euclidean space.

**Definition 5.2** (Rectifiable set.). A set  $S \subset \mathbb{R}^n$  is called  $(\mathcal{H}^m, m)$ -*rectifiable* (or just *m-rectifiable*) if  $\mathcal{H}^m(S) < \infty$  and there is a family  $U_j \subset \mathbb{R}^m$  of bounded subsets and Lipschitz maps  $f_j : U_j \rightarrow \mathbb{R}^n$ ,  $j = 1, 2, \dots$ , so that

$$\mathcal{H}^m \left( S \setminus \bigcup_{j=1}^{\infty} f_j(U_j) \right) = 0.$$

Now we will be doing a sort of differential geometry on these sets, so we will need the notion of a tangent space. Specifically, we define a *tangent cone* (a *cone* being a subset  $C$  of a vector space  $V$  so that  $rc \in C$  for all  $r \in \mathbb{R}_+$  and  $c \in C$ ) in the following manner. If  $V$  is a vector space and  $S \subset V$ , then for  $a \in V$ ,

$$\text{Tan}(S, a) = \{v \in V : \forall \epsilon > 0, \text{ there is an } x \in S \text{ and } r > 0$$

$$\text{s.t. } |x - a| < \epsilon \text{ and } |r(x - a) - v| < \epsilon\}.$$

Now  $\text{Tan}(S, a)$  may be a union of sets of different dimensions. In order to have only the  $m$ -dimensional pieces left, we additionally define the *density* of a measure at a point. Recall that for a measure  $\varphi$  on  $\mathbb{R}^n$ , and a  $\varphi$ -measurable set  $E \subset \mathbb{R}^n$ , the measure  $\varphi \lfloor E$  is defined by

$$(\varphi \lfloor E)(U) := \varphi(E \cap U)$$

for  $\varphi$ -measurable  $U \subset \mathbb{R}^n$ . Then we define the  $m$ -dimensional densities

$$\Theta^m(\varphi, a) := \lim_{r \rightarrow 0} \frac{\varphi(\mathbb{B}(a, r))}{\alpha_m r^m},$$



and

$$\begin{aligned}\Theta^m(E, a) &:= \Theta^m(\mathcal{H}^m \llcorner E, a) \\ &= \lim_{r \rightarrow 0} \frac{\mathcal{H}^m(E \cap \mathbb{B}(a, r))}{\alpha_m r^m}.\end{aligned}$$

Now we may define for a subset  $E \subset \mathbb{R}^n$  and a point  $a \in E$  the *approximate tangent cone of  $E$  at  $a$*  by

$$\text{Tan}^m(E, a) := \bigcap_S \{ \text{Tan}(S, a) : \Theta^m(E \setminus S, a) = 0 \}.$$

Intuitively, the tangent cones that “look like” an  $m$ -dimensional plane will have nonzero density at  $a$ . The point of discussing these rectifiable sets is the following property:

**Theorem 5.3** ([Fed69], 3.2.19). *If  $\Omega$  is an  $m$ -rectifiable and  $\mathcal{H}^m$  measurable subset of  $\mathbb{R}^n$ , then for  $\mathcal{H}^m$  almost all  $a \in \Omega$ ,*

$$\Theta^m(\Omega, a) = 1$$

*and  $\text{Tan}^m(\Omega, a)$  is an  $m$ -dimensional subspace of  $\mathbb{R}^n$ .*

We use this later to provide the existence of orienting vector fields for  $m$ -rectifiable subsets of  $\mathbb{R}^n$ .

#### 5.1.4 Integer multiplicity rectifiable currents.

We define the notion of an integer multiplicity rectifiable  $m$ -current, which is an  $m$ -rectifiable set, together with an orientation and (integer) multiplicity.

**Definition 5.4** (Integer multiplicity rectifiable current.). Let  $1 \leq m \leq n$  be integers, and  $T \in \mathcal{D}_m(\Omega)$ , for an open set  $\Omega \subset \mathbb{R}^n$ . We say  $T$  is an *integer multiplicity rectifiable  $m$ -current* (or just *rectifiable current*) if

$$T(\omega) = \int_S \langle \omega(x), \vec{S}(x) \rangle \theta(x) d\mathcal{H}^m(x)$$

for  $\omega \in \mathcal{D}^m(\Omega)$ , where

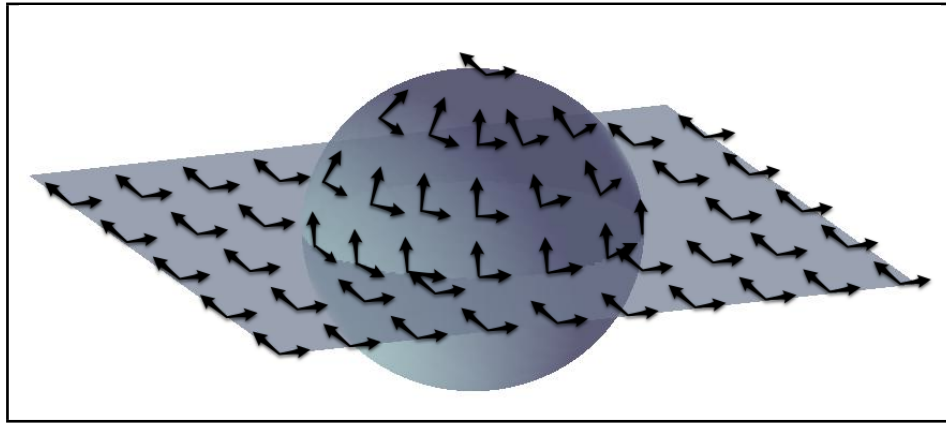
1. **Carrying set.**  $S \subset \Omega$  is  $\mathcal{H}^m$ -measurable and countably  $m$ -rectifiable;
2. **Orientation.**  $\vec{S} : S \rightarrow \bigwedge_m(\mathbb{R}^n)$  is an  $\mathcal{H}^m$ -measurable function such that  $\vec{S}(x)$  is a simple unit  $M$ -vector in  $\mathbf{T}_x S$  for  $\mathcal{H}^m$ -almost every  $x \in S$ ; and
3. **Multiplicity.**  $\theta : S \rightarrow \mathbb{Z}_+$  is locally  $\mathcal{H}^m$ -integrable.

We will write in this situation  $T = \tau(S, \theta, \vec{S})$ . Note that even if a current is supported on a rectifiable set, it may fail to be integer multiplicity rectifiable if the orientation does not lie in the set of tangent vectors almost everywhere. See Figures 5.1 and 5.2.

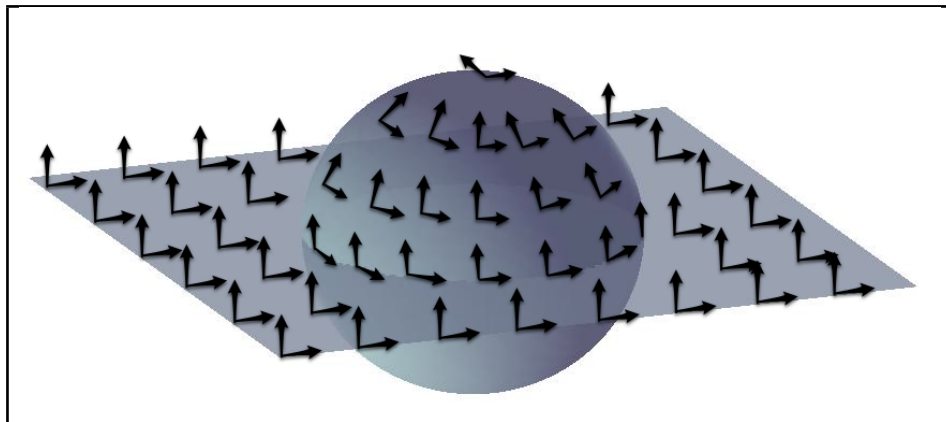
As a subset of the dual space  $\mathcal{D}_m(\Omega)$ , the Banach-Alaoglu theorem gives weak-\* compactness of the unit  $\mathbb{M}$ -ball, but this only gives convergence of a sequence of integer multiplicity rectifiable currents to a general current in  $\mathcal{D}_m(\Omega)$ , rather than another integer multiplicity rectifiable current. In fact, this stronger convergence *does* hold, due to this celebrated result of Federer and Fleming:

**Theorem 5.5** ([FF60]). *Let  $\{T_j\}_{j=1}^\infty \subset \mathcal{D}_m(\Omega)$  be a sequence of integer-multiplicity currents such that*

$$\sup_{j \geq 1} \mathbb{M}(T_j) + \mathbb{M}(\partial T_j) < \infty.$$



**Figure 5.1:** The integer multiplicity 2-current associated with the union of a sphere and a plane in  $\mathbb{R}^3$ , along with an indication of the orienting vector field, which lies in the tangent set. It may be interesting to note that the tangent cone along the intersection of the sets is the union of two planes, so there is a choice in the orienting vector field there.



**Figure 5.2:** This set, along with the indicated orientation is *not* an integer rectifiable 2-current, since the orienting vector field does not lie in the tangent set. Specifically, the orienting planes “stick out” out the set.

Then there is an integer-multiplicity current  $T \in \mathcal{D}_m(\Omega)$  and a subsequence  $\{T_{j'}\}$  such that  $T_{j'} \rightarrow T$  weakly in  $\Omega$ .

### 5.1.5 Euclidean flat chains.

We wish to define the slice of a current, and the proper subset of the space of  $m$ -currents on which to define this are called *flat chains*. Recall the definition of the seminorms on differential forms  $\phi \in \mathcal{E}^m(U)$ ,  $U \subset \mathbb{R}^{m+k}$ ,  $k \in \mathbb{Z}_+$ , given by Equation 5.1, with  $j = 0$  and a compact  $K \subset U$ :

$$\nu_K^0(\phi) = \sup\{\|\phi(x)\| : x \in K\}.$$

Then we may define the *flat seminorm* on  $\mathcal{D}^m(U)$  by

$$\mathbb{F}_K(\phi) := \sup\{\nu_K^0(\phi), \nu_K^0(d\phi)\}.$$

In turn, the dual flat seminorm (defined for *any* real valued linear function  $T$  on  $\mathcal{D}^m(U)$ , not just those continuous  $T \in \mathcal{D}_m(U)$ ) is given by

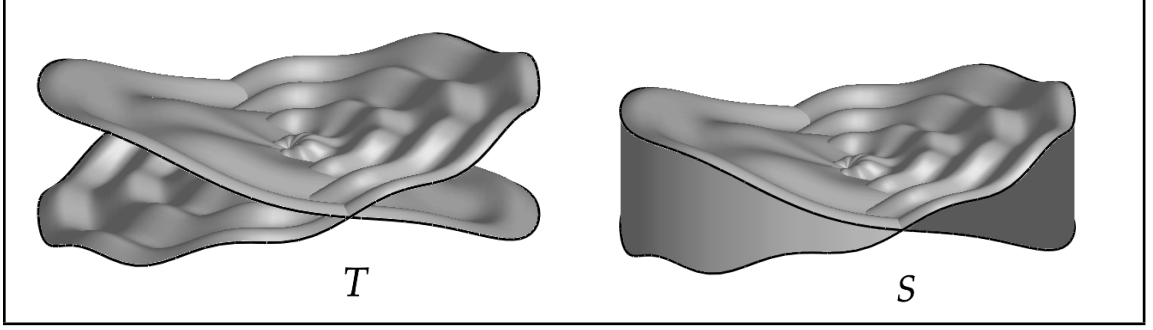
$$\mathbb{F}_K(T) := \sup\{T(\phi) : \phi \in \mathcal{D}^m(U), \mathbb{F}_K(\phi) \leq 1\}.$$

has the following properties:

**Theorem 5.6** ([Fed69], 4.1.12). *For  $U$  as above, and  $T$  a real valued linear function on  $\mathcal{D}^n(U)$ ,*

1. *if  $\mathbb{F}_K(T) < \infty$ , then  $T \in \mathcal{D}_m(U)$  with  $\text{spt } T \subset K$ ,*

2.  *$\mathcal{D}_m(U) \cap \{T : \mathbb{F}_K(T) < \infty\}$  is  $\mathbb{F}_K$  complete,*



**Figure 5.3:** Rectifiable currents  $T \in \mathcal{D}_2(\mathbb{R}^3)$  and  $S \in \mathcal{D}_3(\mathbb{R}^3)$ . Since  $\text{spt } T \subset \text{spt}(\partial S)$ , we have that  $\mathbb{F}_K(T) \leq \mathbb{M}(T - \partial S) + \mathbb{M}(S)$ , which is the sum of the three-dimensional volume of  $S$  and the two-dimensional surface area of the “edge” of  $S$ , for all compact  $K$  containing the support of  $T$ .

3. if  $T \in \mathcal{D}_m(U)$ , then  $\mathbb{F}_K(\partial T) \leq \mathbb{F}_K(T)$ , and

$$\mathbb{F}_{K \cap \text{spt } \gamma}(T \llcorner \gamma) \leq (\mathbf{v}_K^0(\gamma) + \mathbf{v}_K^0(d\gamma))\mathbb{F}_K(T)$$

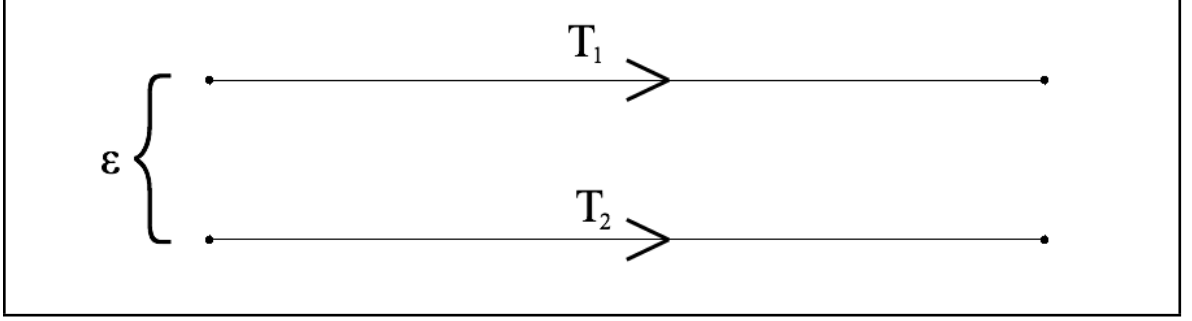
whenever  $\gamma \in \mathcal{D}^0(U)$ , and

4. if  $T \in \mathcal{D}_m(U)$  with  $\text{spt } T \subset K$ ,

$$\mathbb{F}_K(T) = \inf\{\mathbb{M}(T - \partial S) + \mathbb{M}(S) : S \in \mathcal{D}_{m+1}(U) \text{ with } \text{spt } S \subset K\}. \quad (5.2)$$

Thus we have the space of flat chains is complete and contained in the space of  $m$ -currents. Equation 5.2 gives an intuitive way of understanding the flat norm of the current induced by a rectifiable set  $T$ , by trying to find a higher dimensional set  $S$  whose boundary contains “most” of  $T$ . See Figure 5.3. It also gives a more geometric notion of when two currents are “close” than the mass norm. See Figure 5.4.

Using the flat seminorms, we define three vector spaces:



**Figure 5.4:** Two currents  $T_1$  and  $T_2$  which are line segments of length 1, separated by  $\epsilon$ .

Then  $\mathbb{M}(T_1 - T_2) = 2$ , but  $\mathbb{F}(T_1 - T_2) \leq 3 \cdot \epsilon$ , so the currents are “close” in the flat norm, but not the mass norm.

**Definition 5.7** (Flat chains.). Let  $U \subset \mathbb{R}^{m+k}$  be open. Then for each compact  $K \subset U$ , we define

- $\mathbb{F}_{m,K}(U)$  is the  $\mathbb{F}_K$  completion of  $\mathbb{N}_{m,K}(U)$  in  $\mathcal{D}_m(U)$ ,
- $\mathbb{F}_m(U)$ , the vector space of  $m$ -dimensional flat chains, is the union over all the  $\mathbb{F}_{m,K}(U)$  corresponding to all compact  $K \subset U$ , and
- $\mathbb{F}_m^{loc}(U)$ , the locally flat  $m$ -chains, is the vector space of all  $T \in \mathcal{D}_m(U)$  so that  $T|_{\gamma} \in \mathbb{F}_m(U)$  for every  $\gamma \in \mathcal{D}^0(U)$ .

We emphasize that flat chains are defined as the *completion* of the normal currents, rather than just the set of all currents for which  $\mathbb{F}_K(T) < \infty$ .

### 5.1.6 The pullback of a form.

The final preliminary definition we need to define the slice of a flat  $m$ -chain is the pullback of a form. Let  $\Omega \subset \mathbb{R}^m$ ,  $f \in \text{Lip}(\Omega, \mathbb{R}^n)$  and  $\phi \in \mathcal{D}^\ell(\mathbb{R}^m)$  be a differential

$\ell$ -form. Then for almost every  $\mathbf{y} \in f(\Omega)$ , the *pullback* of  $\phi$  at  $\mathbf{y}$  is the  $\ell$ -form denoted by  $f^\# \phi$ , and defined by

$$\langle f^\# \phi(\mathbf{y}), \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\ell \rangle = \langle \phi(f(\mathbf{y})), D_{\mathbf{v}_1} f \wedge \dots \wedge D_{\mathbf{v}_\ell} f \rangle$$

for any  $\ell$ -vector  $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\ell$ , and where  $D_{\mathbf{v}_j} f := Df \mathbf{v}_j$  is the directional derivative of  $f$  in the direction  $\mathbf{v}_j$ .

### 5.1.7 Slicing.

Now we have the background to define the *slice of a flat chain*  $T \in \mathbb{F}_{\ell,K}(U)$ , in  $f^{-1}(\mathbf{y})$ , where  $U \subset \mathbb{R}^m$  is open,  $f : U \rightarrow \mathbb{R}^n$  is locally Lipschitz,  $K \subset U$  is compact and  $m \geq \ell \geq n$ . We will be interested in slices of integer multiplicity rectifiable currents  $T = \tau(S, \theta, \vec{S})$  where  $\mathcal{H}^m(S) < \infty$ , so that  $T$  is actually a normal current.

In particular, we will define, for  $\mathcal{L}^n$ -almost every  $\mathbf{y} \in \mathbb{R}^n$ , an  $\ell - n$  dimensional current  $\langle T, f, \mathbf{y} \rangle$ , which will be called the *slice of  $T$  in  $f^{-1}(\mathbf{y})$* . This current  $\langle T, f, \mathbf{y} \rangle$  will be a limit of currents  $T \llcorner f^\# \phi_j$  corresponding to  $\phi_j \in \mathcal{D}^n(\mathbb{R}^n)$  whose support tends to  $\mathbf{y}$  and whose integral is one. This gives, for currents induced by a rectifiable set  $S$ , the intersection of  $S$  with the level sets of  $f$ ,  $f^{-1}(\mathbf{y})$ .

In the general case, we have from Federer that for  $f$  as above and  $\phi$  a Baire form of degree  $n$  on  $\mathbb{R}^n$ , there is an  $\mathcal{L}^n$ -summable  $n$ -vector field  $\xi_\phi$  on  $\mathbb{R}^n$  so that

$$f_\#(T \llcorner \phi) = \mathcal{L}^n \wedge \xi_\phi.$$

Then we may define  $T \llcorner f^\#(\phi)$  even for these general  $f$  and  $\phi$  by the following calcu-

lation, where we assume first more smoothness for  $\phi$ :

$$\begin{aligned}
\int \langle \xi_\psi, \phi \rangle d\mathcal{L}^n &= f_\#(T \lfloor \psi)(\phi) \\
&= (T \lfloor \psi)(f^\# \phi) \\
&= T(\psi \wedge f^\# \phi) \\
&= (-1)^{n(m-n)} T(f^\# \phi \wedge \psi) \\
&= (-1)^{n(m-n)} (T \lfloor f^\# \phi)(\psi).
\end{aligned}$$

This suggests defining, for general  $\phi$ , a weak version of  $T \lfloor f^\# \phi$  by

$$(T \lfloor f^\# \phi)(\psi) := (-1)^{n(m-n)} \int \langle \xi_\psi, \phi \rangle d\mathcal{L}^n.$$

The following proposition summarizes the regularity and convergence of this current for sequences  $\phi_j$ ,  $T_j$  and  $f_j$ .

**Proposition 5.8.** *Assume  $T$ ,  $\phi$  and  $f$  are as above.*

- **Regularity.** *If  $T \in \mathbb{F}_{m,K}(U)$ , then  $T \lfloor f^\# \phi \in \mathbb{F}_{m-n,K}(U)$ .*
- **Convergence of  $\phi_j$ 's.** *If  $\phi_j$  are Baire forms of degree  $n$  on  $\mathbb{R}^n$  such that*

$$\lim_{j \rightarrow \infty} \phi_j(\mathbf{y}) = \phi(\mathbf{y})$$

*for  $\mathbf{y} \in f(K)$ , and*

$$\sup\{\|\phi_j(\mathbf{y})\| : \mathbf{y} \in f(K), j \in \mathbb{N}\} < \infty,$$

*then*

$$\lim_{j \rightarrow \infty} \mathbb{F}_K((T \lfloor f^\# \phi_j) - (T \lfloor f^\# \phi)) = 0.$$



- **Convergence of  $T_j$ 's.** If also  $T_j \in \mathbb{F}_{m,K}(U)$  for each  $j \in \mathbb{N}$  and  $\mathbb{F}_K(T_j - T) \rightarrow 0$  as  $j \rightarrow \infty$ , then

$$\lim_{j \rightarrow \infty} \mathbb{F}_K(T_j \llcorner f^\# \phi - T \llcorner f^\# \phi) = 0.$$

- **Convergence of  $f_j$ 's.** If  $f_j \in C^\infty(U; \mathbb{R}^n)$  with  $\lim_{j \rightarrow \infty} \|f_j - f\|_{L^\infty(K; \mathbb{R}^n)} = 0$  and  $\sup_j \|Df_j\|_{L^\infty(K; \mathbb{R}^n)} < \infty$ , then

$$\lim_{j \rightarrow \infty} \mathbb{F}_K(T \llcorner f_j^\# \phi - T \llcorner f^\# \phi) = 0.$$

Now we may finally define the slice of a current  $T \in \mathbb{F}_m(U)$  by a Lipschitz  $f : U \rightarrow \mathbb{R}^n$  at  $\mathbf{y} \in \mathbb{R}^n$ , denoted  $\langle T, f, \mathbf{y} \rangle \in \mathcal{D}_{m-n}(U)$  as the limit

$$\langle T, f, \mathbf{y} \rangle(\psi) := \lim_{r \rightarrow 0} (T \llcorner f^\# \zeta_{r, \mathbf{y}})(\psi).$$

where  $\zeta_{r, \mathbf{y}} : \mathbb{R}^n \rightarrow \bigwedge_n \mathbb{R}^n$  is defined by

$$\zeta_{r, \mathbf{y}}(\mathbf{x}) = \frac{\chi_{\mathbb{B}(\mathbf{y}, r)}(\mathbf{x}) \wedge dx_1 \wedge \cdots \wedge dx_n}{\alpha_n r^n}$$

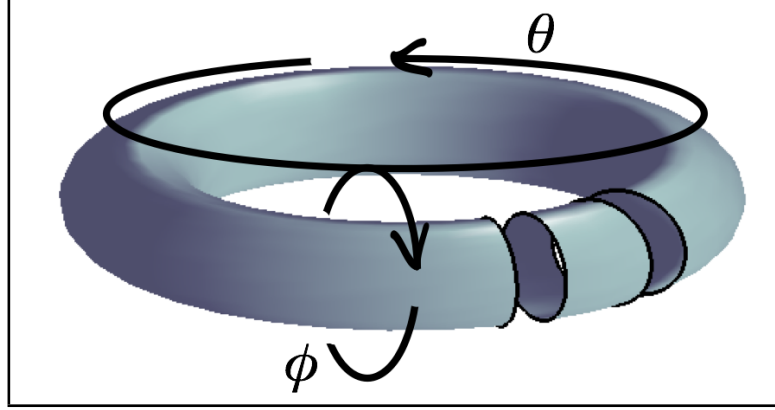
See Figure 5.5 for an example of slicing a rectifiable current, and Federer [Fed69] 4.3.1, for details on the proof of convergence of this limit (i.e., existence of this current).

### 5.1.8 Existence.

We require the following lemma to show that the limit of the slices agrees with the limits of the functions, at least in the flat norm.

**Lemma 5.9** (Convergence of slices.). *Suppose  $f_j \in \text{Lip}_\ell(\Omega; \mathbb{R}^n)$ , where  $\Omega \subset \mathbb{R}^m$  is open,  $m \geq n$ ,  $f_j \rightarrow f$  in the sup norm, and  $T \in \mathbb{N}(\Omega)$  is a normal current. Then*

$$\lim_{j \rightarrow \infty} \mathbb{F}_K(\langle T, f_j, y \rangle - \langle T, f, y \rangle) = 0$$



**Figure 5.5:** If  $T$  is the rectifiable current associated with the torus (the orientation is not shown), and  $f : \text{supp } T \rightarrow \mathbb{R}$  is given by  $f(\theta, \phi) = \theta$ , where  $\text{supp } T$  is parameterized as indicated, then the disconnected section of the torus is the rectifiable current  $T \llcorner f^\# \xi_{r, \theta_0}$ , for some small value of  $r$ . As  $r \rightarrow 0$ , this current will converge to the slice of the torus as  $\theta_0$ , a circle.

for each compact  $K \subset \Omega$  and almost every  $y \in \mathbb{R}^n$ .

**Remark 5.10.** Recall that since  $T \in \mathbb{N}(\Omega)$ , then in particular  $T \in \mathbb{F}(\Omega)$ , since  $\mathbb{F}_K(T) \leq \mathbb{M}_K(T) \leq \mathbb{N}_K(T) = \mathbb{M}_K(T) + \mathbb{M}_K(\partial T)$ . In particular, we may let  $T = \tau(\Omega, \theta, \vec{\Omega})$  be the integer multiplicity rectifiable current associated with some rectifiable set  $\Omega \subset \mathbb{R}^m$  so long as  $\mathcal{H}^m(\Omega) + \mathcal{H}^{m-1}(\partial\Omega) < \infty$ .

*Proof.* We have from [Fed69], 4.3.9 the slicing homotopy formula, which gives that if  $f, g \in \text{Lip}_\ell(\Omega; \mathbb{R}^n)$ ,  $T \in \mathbb{N}(\Omega)$ ,  $E \subset \Omega$  is a Borel set, and  $h : [0, 1] \times \Omega \rightarrow \mathbb{R}^n$  is the affine homotopy

$$h(t, x) = h_t(x) = (1 - t)f(x) + tg(x),$$

we have

$$\int_E \mathbb{F}(\langle T, g, y \rangle - \langle T, f, y \rangle) d\mathcal{L}^n(y) \leq \ell^{n-1} \int_0^1 \int_{h_t^{-1}} |g - f| d(\mu_t + \mu_{\partial T}) d\mathcal{L}^1(t). \quad (5.3)$$

Applying Eq. 5.3 to our situation, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbb{F}(\langle T, f_j, y \rangle - \langle T, f, y \rangle) d\mathcal{L}^n(y) &\leq \ell^{n-1} \int_0^1 \int_{h_t^{-1}} |f_j - f| d(\mu_t + \mu_{\partial T}) d\mathcal{L}^1(t) \\ &\leq \ell^{n-1} \|f_j - f\|_{L^\infty(\Omega; \mathbb{R}^n)} \mathbb{N}(T) < \infty. \end{aligned} \quad (5.4)$$

Now extract a subsequence, not relabeled,  $\{f_j\}$  with the property that  $\|f - f_j\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 2^{-j}$  for each  $j = 1, 2, \dots$ . Using Lebesgue's monotone convergence theorem, we find that

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} \mathbb{F}(\langle T, f_j, y \rangle - \langle T, f, y \rangle) d\mathcal{L}^n(y) &= \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \mathbb{F}(\langle T, f_j, y \rangle - \langle T, f, y \rangle) d\mathcal{L}^n(y) \\ &\leq \ell^{n-1} \mathbb{N}(T) \sum_{j=1}^{\infty} \|f_j - f\|_{L^\infty(\Omega; \mathbb{R}^n)} \\ &= \frac{\ell^{n-1} \mathbb{N}(T)}{2}. \end{aligned}$$

From this convergence, we may conclude that  $\sum_{j=1}^{\infty} \mathbb{F}(\langle T, f_j, y \rangle - \langle T, f, y \rangle)$  is finite for almost every  $y \in \mathbb{R}^n$ , and in particular, that

$$\lim_{j \rightarrow \infty} \mathbb{F}(\langle T, f_j, y \rangle - \langle T, f, y \rangle) = 0$$

for almost every  $y \in \mathbb{R}^n$ . □

**Theorem 5.11** (Existence). *Suppose  $\Omega \subset \mathbb{R}^m$  is rectifiable and bounded,  $\ell \geq 0$ ,  $h \in \text{Lip}_\ell(\overline{\Omega}; \mathbb{R}^n)$ . Then there exists an  $f \in \text{Lip}_\ell(\overline{\Omega}, \mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f^{-1}(y) \cap \Omega) d\mathcal{H}^n(y) \leq \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(g^{-1}(y) \cap \Omega) d\mathcal{H}^n(y)$$

for all  $g \in Lip_\ell(\overline{\Omega}, \mathbb{R}^n)$  with

$$\partial \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), g, y \rangle = \partial \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), h, y \rangle,$$

where  $\overrightarrow{\Omega}$  is an orienting vector field for  $\Omega$ .

*Proof.* First, we will denote the set of admissible functions by

$$\mathcal{A} := \left\{ g \in Lip_\ell(\overline{\Omega}, \mathbb{R}^n), \partial \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), g, y \rangle = \partial \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), h, y \rangle \right\}$$

notice that by Ascoli-Arzelà,  $Lip_\ell(\overline{\Omega}, \mathbb{R}^n)$  (and so  $\mathcal{A}$ ) is compact, so if we let

$$M := \inf_{\mathcal{A}} \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(g^{-1}(y) \cap \Omega) d\mathcal{H}^n(y),$$

then there is a sequence  $\{f_j\} \in \mathcal{A}$  with

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f_j^{-1}(y) \cap \Omega) d\mathcal{H}^n(y) = M.$$

Using Theorem 3.2.22 and 4.3.8 from [Fed69], we have that  $f_j^{-1}(y)$  is rectifiable for almost every  $y$ , and so

$$\int_{\mathbb{R}^n} \mathbb{M}(\langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f, y \rangle) d\mathcal{L}^n(y) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f_j^{-1}(y) \cap \Omega) d\mathcal{L}^n(y). \quad (5.5)$$

We also have convergence of the slices: applying Lemma 5.9, for almost every  $y$ ,  $\lim_{j \rightarrow \infty} \mathbb{F}_K \left( \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f_j, y \rangle - \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f, y \rangle \right) = 0$ . This convergence in the flat norm implies convergence in the weak norm, since for all smooth forms  $\psi$  with  $\mathbb{M}(\psi) \leq 1$ ,

$$\begin{aligned} & \left( \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f, y \rangle - \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f_j, y \rangle \right) (\psi) \\ & \leq \mathbb{F}_K \left( \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f, y \rangle - \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f_j, y \rangle \right) \mathbb{F}_K(\psi) \\ & \leq \mathbb{F}_K \left( \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f, y \rangle - \langle \tau(\Omega, \theta, \overrightarrow{\Omega}), f_j, y \rangle \right), \end{aligned}$$

using also that  $\mathbb{F}_K(\psi) \leq \mathbb{M}(\psi)$ .

In turn, recall that weak convergence implies lower semicontinuity of mass as follows: for any  $\epsilon > 0$ , there exists a smooth form  $\psi$  so that

$$\begin{aligned} \mathbb{M} \left( \langle \tau(\Omega, \theta, \vec{\Omega}), f, y \rangle \right) &\leq \epsilon + \langle \tau(\Omega, \theta, \vec{\Omega}), f, y \rangle(\psi) \\ &= \epsilon + \lim_{j \rightarrow \infty} \langle \tau(\Omega, \theta, \vec{\Omega}), f_j, y \rangle(\psi) \\ &\leq \epsilon + \liminf_{j \rightarrow \infty} \mathbb{M} \left( \langle \tau(\Omega, \theta, \vec{\Omega}), f_j, y \rangle \right). \end{aligned}$$

Applying this last estimate, as well as Fatou's lemma, gives us the desired inequalities:

$$\begin{aligned} M &\leq \int \mathbb{M} \left( \langle \tau(\Omega, \theta, \vec{\Omega}), f, y \rangle \right) \\ &\leq \int \lim_{j \rightarrow \infty} \mathbb{M} \left( \langle \tau(\Omega, \theta, \vec{\Omega}), f_j, y \rangle \right) \\ &\leq \liminf_{j \rightarrow \infty} \int \mathbb{M} \left( \langle \tau(\Omega, \theta, \vec{\Omega}), f_j, y \rangle \right) \\ &= M. \end{aligned}$$

So  $f$  is the desired minimizer. □

Notice that the above proof required only that  $m \geq n$ , and did not require that the current being sliced was actually integer multiplicity rectifiable, only normal. In particular, we have the following, more general result:

**Theorem 5.12** (Minimizers for normal currents.). *Suppose  $U \subset \mathbb{R}^m$  is open,  $T \in \mathbb{N}_m(U)$ ,  $\ell \geq 0$ ,  $h \in Lip_\ell(U; \mathbb{R}^n)$ . Then for any compact  $K \subset U$ , there exists an  $f \in Lip_\ell(U, \mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} \mathbb{M}_K(\langle T, f, y \rangle) d\mathcal{L}^n(y) \leq \int_{\mathbb{R}^n} \mathbb{M}_K(\langle T, g, y \rangle) d\mathcal{L}^n(y)$$

for all  $g \in Lip_\ell(U, \mathbb{R}^n)$  with

$$\partial \langle T, g, y \rangle = \partial \langle T, h, y \rangle \text{ for a.e. } y \in \mathbb{R}^n.$$

# Chapter 6

## The coarea formula in metric spaces.

### 6.1 An approach using metric currents.

We follow the work of De Pauw and Hardt [DPH12] to place the question of minimizers in the context of rectifiable chains. As is usual in the study of analysis in metric spaces, we will rely on Lipschitz maps and Hausdorff measure, as each requires only a concept of distance, not direction, to define.

In the sequel, we will let  $X$  be a metric space, and  $M \subset X$  be an  $\mathcal{H}^m$ -rectifiable subset of  $X$ . Note that in the context of a metric space  $X$ , the definition of rectifiable is the same as that given in Definition 5.2 with  $\mathbb{R}^n$  replaced with  $X$ :

**Definition 6.1** (Rectifiable set in a metric space). A set  $S \subset X$  is called  $(\mathcal{H}^m, m)$ -*rectifiable* (or just *m-rectifiable*) if  $\mathcal{H}^m(S) < \infty$  and there is a family  $A_j \subset \mathbb{R}^m$  of

bounded subsets and Lipschitz maps  $f_j : A_j \rightarrow X$ ,  $j = 1, 2, \dots$ , so that

$$\mathcal{H}^m \left( S \setminus \bigcup_{j=1}^{\infty} f_j(A_j) \right) = 0.$$

### 6.1.1 Preliminaries.

First, we define a notion of parametrization of  $M$ :

**Definition 6.2** (Locally bilipschitz almost parameterization of  $M$ ). For an  $\mathcal{H}^m$ -rectifiable subset  $M$  of a metric space  $X$ , we say the pair  $(\gamma, \{A_k\}_{k=1}^{\infty})$  is a *locally bilipschitz almost parameterization of  $M$*  provided

1. each  $A_k \subset \mathbb{R}^m$  is compact,
2. the  $A_k$ 's are disjoint,
3. the restricted maps  $\gamma|_{A_k}$  are bilipschitz for each  $k$  with disjoint images in  $M$ ,  
and
4.  $\mathcal{H}^m(M \setminus \gamma(\cup_{k=1}^{\infty} A_k)) = 0$ .

As a corollary of Lemma 3.1.1 of [DPH12], as well as the discussion afterwards, we note that every  $\mathbb{R}^m$  rectifiable subset  $M$  has a locally bilipschitz almost parameterization. Intuitively, one takes the pair  $\{f_j, A_j\}$  from Definition 6.1 and translate the (bounded) sets  $A_j$  so that they are disjoint, then defining the function  $\gamma$  on each set via these original functions:  $\gamma|_{A_j} := f_j$ .

We additionally recall the definition and some results of rectifiable  $\mathbb{Z}$ -chains over a rectifiable subset of a metric space. De Pauw and Hardt use the strategy of defining



such chains as the mass norm completion of a certain collection of chains, though their coefficients are in a normed abelian group  $G$ . As mentioned, when the coefficients are  $\mathbb{R}$  or  $\mathbb{Z}$ , this is the same set as Ambrosio and Kirchheim construct in [AK00] or Lang in [Lan11], respectively.

For  $a \in X$  and  $k \in \mathbb{Z}$ , we let  $k\llbracket a \rrbracket$  denote the atomic measure

$$(k\llbracket a \rrbracket)(E) = \begin{cases} k & \text{if } a \in E, \\ 0 & \text{otherwise,} \end{cases}$$

for  $E \subset X$ . Then further we define the space of finite 0-dimensional chains,

$$\mathcal{L}_0(X; \mathbb{Z}) = \left\{ \sum_{a \in A} k_a \llbracket a \rrbracket : A \subset X \text{ is finite, } k_a \in \mathbb{Z} \right\},$$

which is equipped with the *mass norm*,

$$\mathbb{M} \left( \sum_{a \in A} k_a \llbracket a \rrbracket \right) = \sum_{a \in A} |k_a|.$$

Then the group  $\mathcal{R}_0(X; \mathbb{Z})$  of 0-dimensional rectifiable  $\mathbb{Z}$  chains,

$$\mathcal{R}_0(X; \mathbb{Z}) = \left\{ \sum_{a \in A} k_a \llbracket a \rrbracket : A \subset X \text{ is countable, } k_a \in \mathbb{Z}, \sum_{a \in A} |k_a| < \infty \right\},$$

is the  $\mathbb{M}$ -completion of  $\mathcal{L}_0(X; \mathbb{Z})$ . In this case we actually have equality:  $\mathcal{R}_0(X; \mathbb{Z}) = \mathcal{L}_0(X; \mathbb{Z})$ . If the coefficients were not integers but instead, for example, in  $\mathbb{R}$ , then this would not be the case. Specifically, if

$$T = \sum_{j=1}^{\infty} \frac{1}{2^j} \llbracket j \rrbracket,$$

then  $T \in \mathcal{R}_0(\mathbb{Z}; \mathbb{R})$ , but  $T \notin \mathcal{L}_0(\mathbb{Z}; \mathbb{R})$ .

Each  $T \in \mathcal{R}_0(X; \mathbb{Z})$  is a purely atomic measure with associated finite positive measure

$$\mu_T(E) := \sum_{a \in A \cap E} |k_a| \text{ for } E \subset X.$$

To describe higher dimensional  $\mathbb{Z}$  chains on the metric space  $X$ , we continue to follow De Pauw and Hardt, and define:

**Definition 6.3** ( $m$ -dimensional parameterized  $\mathbb{Z}$  chain.). An  $m$ -dimensional parameterized  $\mathbb{R}^n$  chain is a triple  $[\gamma, \{A_k\}_{k=1}^\infty, g]$  where

1.  $A_k \subset \mathbb{R}^m$  for each  $k$ , and the double  $(\gamma, \{A_k\}_{k=1}^\infty)$  is a locally bilipschitz almost parametrization,
2.  $g : \cup_{k=1}^\infty A_k \rightarrow \mathbb{Z}$  is measurable, and

3.

$$\sum_{k=1}^\infty \int_{\gamma(A_k)} |g \circ \gamma_k^{-1}| d\mathcal{H}^m < \infty,$$

where  $\gamma_k := \gamma|_{A_k}$  is the restriction of  $\gamma$  to  $A_k$ .

We define the equivalence relation  $\approx$  on parameterized  $\mathbb{Z}$ -chains as follows. We say  $[\gamma, \{A_k\}, g] \approx [\tilde{\gamma}, \{\tilde{A}_k\}, \tilde{g}]$  if the  $m$ -dimensional Hausdorff measure of the symmetric difference of the parameterizations is zero:

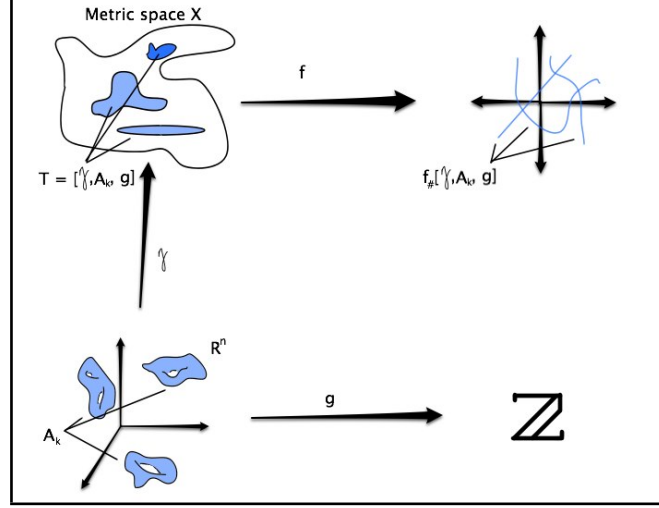
$$\mathcal{H}^m \left( \gamma(\cup_{j=1}^\infty A_j) \Delta \tilde{\gamma}(\cup_{j=1}^\infty \tilde{A}_j) \right) = 0,$$

and up to orientation, the maps  $\gamma_j := \gamma|_{A_j}$  are coherent almost everywhere. Specifically, if  $\gamma(A_j) \cap \tilde{\gamma}_k(\tilde{A}_k) \neq \emptyset$ , then

$$g \circ \gamma_j^{-1} = \sigma_{j,k} \cdot (\tilde{g} \circ \tilde{\gamma}_k^{-1}),$$

where  $\sigma_{j,k}$  keeps track of a change of orientation:

$$\sigma_{j,k} := \text{sgn det} \left( D(\gamma_j^{-1} \circ \tilde{\gamma}_k) \circ \tilde{\gamma}_k^{-1} \right) = \text{sgn det} \left( D(\tilde{\gamma}_k^{-1} \circ \gamma_j) \circ \gamma_j^{-1} \right).$$



**Figure 6.1:** A rough sketch of the notation used, and relationship between the parametrization  $\gamma$ , coefficients in  $\mathbb{Z}$  given by  $g$ , sets  $A_k$ , and push-forward map  $f$ . One of the regions  $\gamma(A_k)$  in  $X$  is colored more darkly in an attempt to indicate a greater multiplicity of  $g$  on  $A_k$ .

Then an equivalence class of  $m$ -dimensional parameterized  $\mathbb{Z}$  chains is called an  $m$ -dimensional *rectifiable*  $\mathbb{Z}$  chain, and we denote this space by  $\mathcal{R}_m(X; \mathbb{Z})$ , and denote an equivalence class by  $[\gamma, \{A_j\}, g]$ .

Our use for such  $m$ -dimensional  $\mathbb{Z}$  chains will be to look at their *push-forward* via a (bilipschitz) map  $f : M \subset X \rightarrow \mathbb{R}^n$ . We may define a version of the coarea formula for such maps via *slices* of the map, the theory of which we develop below.

**Definition 6.4** (Bilipschitz push-forward). Suppose  $X, Y$  are metric spaces,  $T = [\gamma, \{A_j\}, g] \in \mathcal{R}_m(X; \mathbb{Z})$ , and  $f : X \rightarrow Y$  is bilipschitz. Then we define the *push-forward* of  $T$  via  $f$

$$f_{\#}[\gamma, \{A_j\}, g] := [f \circ \gamma, \{A_j\}, g] \in \mathcal{R}_m(Y; \mathbb{Z}).$$

Notice that  $f_{\#}[\![\gamma, \{A_j\}, g]\!]$  can be seen to be well-defined via the equivalence relation  $\approx$ .

To extend the push-forward to Lipschitz maps (rather than only bilipschitz ones), we again start with 0-chains, and move up to general rectifiable  $\mathbb{Z}$ -chains. In particular, for a *Lipschitz*  $f$  and countable set  $A \subset X$ , define the push-forward of a 0-chain by

$$f_{\#} \left( \sum_{a \in A} k_a [\![a]\!] \right) := \sum_{a \in A} k_a [\![f(a)]\!].$$

Now let  $T = [\![\gamma, A_k, g]\!] \in \mathcal{R}_m(X; \mathbb{Z})$  be an  $m$ -chain, and we again apply [DPH12] Lemma 3.1.1 to deduce the existence of a disjoint, compact subsets  $\{K_j\}$  of  $A = \cup A_k$  so that  $(f \circ \gamma)|_{K_j}$  is bilipschitz. Then the push-forward is defined by

$$\phi_{\#} T := \sum_{j=1}^{\infty} [(f \circ \gamma)|_{K_j}]_{\#} [\![I_m, K_j, g]\!] = \sum_{j=1}^{\infty} [\![(f \circ \gamma)|_{K_j}, K_j, g]\!],$$

where  $I_m$  is the identity map, and the sum is over previously defined bilipschitz push-forwards. The push-forward  $f_{\#} T$  is independent of both the representation  $[\![\gamma, \{A_k\}, g]\!]$  and the sets  $K_j$ .

We also wish to define the density associated with these chains, as well as some notion of support.

**Definition 6.5** (Density). For  $T = [\![\gamma, \{A_j\}, g]\!] \in \mathcal{R}_m(X; \mathbb{Z})$ , define  $\theta_T : X \rightarrow \mathbb{R}^+$  by

$$\theta_T(x) = \begin{cases} |g(\gamma_k^{-1}(x))| & \text{if } x \in \gamma_k(A_k) \\ 0 & \text{if } x \in X \setminus \cup_{k=1}^{\infty} A_k \end{cases}.$$

Note that by the definition of the  $\gamma$  functions, this is well-defined  $\mathcal{H}^m$  a.e. and is  $\mathcal{H}^m$  integrable. Then we may define the carrying set of  $T$  by

$$M_T = \{x \in X : \theta_T(x) > 0\},$$

which is an  $\mathcal{H}^m$  rectifiable subset of  $X$ , and well-defined up to a set of  $\mathcal{H}^m$  measure 0, as well as the Borel regular measure  $\mu_T$  on  $X$  via integration against  $\theta_T$ :

$$\mu_T(E) = \int_E \theta_T(x) d\mathcal{H}^m(x)$$

for each  $\mathcal{H}^m$  measurable  $E \subset X$ . Finally, the *mass*  $\mathbb{M} : \mathcal{R}_M(X; \mathbb{Z}) \rightarrow \mathbb{R}+$  is defined by

$$\mathbb{M}(T) := \mu_T(X).$$

However, this mass is not known to be lower semicontinuous, so we will define a different notion of mass  $\hat{\mathbb{M}}$  in the following section, which will use the concept of slicing.

We will also need the flat norm in metric spaces. The following is a theorem presented in [DPH12], but maybe be taken as a definition for our purposes:

**Theorem 6.6** (Flat norm.). *For  $T \in \mathcal{R}_m(X; \mathbb{Z})$ , the flat norm is given by*

$$\mathbb{F}(T) := \inf \{ \mathbb{M}(R) + \mathbb{M}(\partial S) \quad : \quad \Phi_{\#}T = R + \partial S, \Phi \text{ is an isometric embedding of } X \text{ into} \\ \text{a Banach space } Y, R \in \mathcal{R}_m(Y; G), S \in \mathcal{R}_{m+1}(Y; G) \}.$$

We justify calling this the flat norm via Theorem 5.6, which gives a similar formula for the Euclidean version of this norm (though notice that in this case we did not go through semi-norms on differential forms first).

## 6.2 Slicing a metric current.

The notion of slicing allows us to discuss the coarea formula in the context of metric spaces. In order to do this, we will recall the notion of a slice from Euclidean space,

and use the  $\mathbb{M}$ -convergent representation from [DPH12]. We will adapt the notation from De Pauw and Hardt and denote, for a rectifiable set  $A_j \subset \mathbb{R}^m$ , the integer rectifiable current  $\tau(A_j, g, \vec{A_j})$  somewhat more compactly as

$$\llbracket A_j, g \rrbracket := \tau(A_j, g, \vec{A_j}).$$

Now, for a flat chain  $T = \llbracket \gamma, A_k, g \rrbracket \in \mathcal{D}_m(X; \mathbb{Z})$ , a Lipschitz  $f : X \rightarrow \mathbb{R}^n$ , and a point  $y \in \mathbb{R}^n$ , we define the slice of  $T$  by  $f$  at  $y$  by

$$\langle T, f, y \rangle = \sum_{j=1}^{\infty} \gamma_{\#} \langle \llbracket A_j, g \rrbracket, f \circ \gamma, y \rangle. \quad (6.1)$$

That is, we are slicing in Euclidean space, then pushing forward along our Lipschitz parametrization, and summing over the countably many disjoint subsets.

Notice also that we get the analogous result to Equation 5.5:

$$\mathbb{M}(\langle T, f, y \rangle) = \int_{M \cap f^{-1}(y)} \theta_T(x) d\mathcal{H}^{m-n}(x), \quad (6.2)$$

which is crucial in discussing the coarea formula in the language of currents.

As mentioned earlier, we may also use slicing to define a mass  $\hat{\mathbb{M}}$ , which is lower-semicontinuous with respect to flat convergence. To do so, let  $m \in \mathbb{Z}_+$ ,  $U \subset X$  open,  $p \in \text{Lip}_1(X, \mathbb{R}^m)$ . Then for any  $T \in \mathcal{R}_m(X; \mathbb{Z})$  and  $\mu_T$ -measurable  $A \subset X$ , define

$$\lambda_{T,U,p}(A) := \int_{\mathbb{R}^m} \mu_{\langle T,p,y \rangle}(U \cap A) dy = \int_{\mathbb{R}^m} \mathbb{M}(\langle T,p,y \rangle \llcorner (U \cap A)) dy.$$

The *slicing mass*  $\hat{\mathbb{M}}(T) := \hat{\mu}_T(X)$  is defined by

$$\hat{\mu}_T(A) := \sup \left\{ \sum_{j=1}^N \lambda_{T,U_j,p_j} : N < \infty, p_j \in \text{Lip}_1(X; \mathbb{R}^m), U_j \subset X \text{ are disjoint, open} \right\}.$$

The following two theorems from De Pauw and Hardt help to motivate our interest in  $\hat{\mathbb{M}}$ :

**Theorem 6.7** ([DPH12],5.7.2).  $\hat{\mathbb{M}}$  is  $\mathbb{F}$ -lower semicontinuous on  $\mathcal{R}_m(X; \mathbb{Z})$ .

**Theorem 6.8** ([DPH12],5.7.4).  $(2m)^{-m}\mathbb{M}(T) \leq \hat{\mathbb{M}}(T) \leq \mathbb{M}(T)$  for any  $T \in \mathcal{R}_m(X; \mathbb{Z})$ .

Note that, while  $\hat{\mathbb{M}} \neq \mathbb{M}$  in general, in certain contexts the two will still be equal. In particular, if  $X = \mathbb{R}^m$  is a Euclidean space, and  $T \in \mathcal{D}_m(X, \mathbb{Z})$  is an integer multiplicity flat  $m$ -chain,  $T = \tau(\Omega, \theta, \vec{\Omega})$ , then choosing  $p$  as the identity map  $Id$ , and again applying Eq. 5.5, we have

$$\begin{aligned} \lambda_{T,U,Id}(A) &= \int_{\mathbb{R}^n} \mathbb{M} \left( \langle \tau(\Omega, \theta, \vec{\Omega}), Id, y \rangle \llcorner (U \cap A) \right) d\mathcal{L}^n(y) \\ &= \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(Id^{-1}(y) \cap (U \cap A)) d\mathcal{L}^m(y) \\ &= \mathcal{L}^m(A \cap U). \end{aligned}$$

It follows that for such an  $m$ -chain  $T$ ,

$$\hat{\mathbb{M}}(T) \geq \mathbb{M}(T).$$

In fact, one may show that  $\hat{\mathbb{M}}(T) = \mathbb{M}(T)$  whenever  $T$  is a 0- or 1-chain, or whenever  $X$  is a Hilbert space.

### 6.3 Existence in metric spaces.

In order to prove the existence of coarea minimizers in this metric space setting, we will need to generalize Lemma 5.9, as follows:

**Lemma 6.9** (Convergence of slices.). *Suppose  $f_j \in \text{Lip}_\ell(X; \mathbb{R}^n)$ ,  $m \geq n$ ,  $f_j \rightarrow f$  in the sup norm, and  $T = \llbracket \gamma, \{A_k\}, g \rrbracket \in \mathcal{R}_m(X; \mathbb{Z})$  is an  $m$ -dimensional rectifiable*

$\mathbb{Z}$ -chain with  $\sum_{k=1}^{\infty} \int_{\gamma(\partial A_k)} |g \circ \gamma_k^{-1}| \, d\mathcal{H}^{m-1} < \infty$ . Then

$$\lim_{j \rightarrow \infty} \mathbb{F}_K (\langle T, f_j, y \rangle - \langle T, f, y \rangle) = 0$$

for each compact  $K \subset \Omega$  and almost every  $y \in \mathbb{R}^n$ .

**Remark 6.10.** *The condition*

$$\sum_{k=1}^{\infty} \int_{\gamma(\partial A_k)} |g \circ \gamma_k^{-1}| \, d\mathcal{H}^{m-1} < \infty$$

is really verifying that the underlying rectifiable set in  $\mathbb{R}^m$  has finite boundary mass, so that we may think of  $T$  as a generalization of a normal current in a Euclidean space.

*Proof.* By Equation 6.1, and [Fed69], 4.1.14, we have

$$\begin{aligned} \mathbb{F}_K (\langle T, f_j, y \rangle - \langle T, f, y \rangle) &= \mathbb{F}_K \left( \sum_{k=1}^{\infty} \gamma_{\#} \langle \llbracket A_k, g \rrbracket, f_j \circ \gamma, y \rangle - \sum_{k=1}^{\infty} \gamma_{\#} \langle \llbracket A_k, g \rrbracket, f \circ \gamma, y \rangle \right) \\ &\leq \sum_{k=1}^{\infty} \mathbb{F}_K (\gamma_{\#} \langle \llbracket A_k, g \rrbracket, f_j \circ \gamma, y \rangle - \gamma_{\#} \langle \llbracket A_k, g \rrbracket, f \circ \gamma, y \rangle) \\ &= \sum_{k=1}^{\infty} \mathbb{F}_K (\gamma_{\#} (\langle \llbracket A_k, g \rrbracket, f_j \circ \gamma, y \rangle - \langle \llbracket A_k, g \rrbracket, f \circ \gamma, y \rangle)) \\ &\leq C \sum_{k=1}^{\infty} \mathbb{F}_K (\langle \llbracket A_k, g \rrbracket, f_j \circ \gamma, y \rangle - \langle \llbracket A_k, g \rrbracket, f \circ \gamma, y \rangle), \end{aligned}$$

where  $C = C(\ell)$  is a constant (specifically,  $C = \sup\{(\text{Lip } f|_K)^n, (\text{Lip } f_K)^{n-1}\}$ ).

Now each summand is bounded by Eq. 5.4:

$$\mathbb{F}_K (\langle \llbracket A_k, g \rrbracket, f_j \circ \gamma, y \rangle - \langle \llbracket A_k, g \rrbracket, f \circ \gamma, y \rangle) \leq \ell^{n-1} \|f_j - f\|_{L^\infty(X; \mathbb{R}^n)} \mathbb{N}(\llbracket A_k, g \rrbracket).$$

Inserting this, and using the hypothesis  $\sum_{k=1}^{\infty} \int_{\gamma(\partial A_k)} |g \circ \gamma_k^{-1}| \, d\mathcal{H}^{m-1} < \infty$  as well as the fact that  $T$  is an  $m$ -dimensional parameterized  $\mathbb{Z}$ -chain, so  $\sum_{k=1}^{\infty} \int_{\gamma(A_k)} |g \circ$



$\gamma_k^{-1}| d\mathcal{H}^m < \infty$ , the last sum above becomes

$$\begin{aligned}
C \sum_{k=1}^{\infty} \mathbb{F}_K(\langle \llbracket A_k, g \rrbracket, f_j \circ \gamma, y \rangle & - \langle \llbracket A_k, g \rrbracket, f \circ \gamma, y \rangle) \\
& \leq C \ell^{n-1} \|f_j - f\|_{L^\infty(X; \mathbb{R}^n)} \sum_{k=1}^{\infty} \mathbb{N}(\llbracket A_k, g \rrbracket) \\
& \leq C(\ell) \|f_j - f\|_{L^\infty(X; \mathbb{R}^n)} \sum_{k=1}^{\infty} \int_{\gamma(\partial A_k)} |g \circ \gamma_k^{-1}| d\mathcal{H}^{m-1} \\
& \quad + \sum_{k=1}^{\infty} \int_{\gamma(A_k)} |g \circ \gamma_k^{-1}| d\mathcal{H}^m,
\end{aligned}$$

where

$$C(\ell) = \ell^{n-1} \sup\{(\text{Lip } f|_K)^n, (\text{Lip } f_K)^{n-1}\}$$

Then as  $j \rightarrow \infty$ , we have our convergence.  $\square$

Now the analogous theorem to Theorems 3.10 and 5.11 provides existence of minimizers of the coarea formula over Lipschitz functions in a metric space. We necessarily use the mass  $\hat{\mathbb{M}}$  for lower semicontinuity, but will stress that it is still an open question whether  $\mathbb{M}$  is lower semicontinuous, in which case the following theorem would hold with this mass, rather than the slicing mass:

**Theorem 6.11** (Minimizers for flat chains.). *Suppose  $X$  is a metric space,  $T = \llbracket \gamma, \{A_k\}, g \rrbracket \in \mathcal{R}_m(X; \mathbb{Z})$  is an  $m$ -dimensional rectifiable  $\mathbb{Z}$ -chain with  $\sum_{k=1}^{\infty} \int_{\gamma(\partial A_k)} |g \circ \gamma_k^{-1}| d\mathcal{H}^{m-1} < \infty$ , and  $h \in \text{Lip}_\ell(X, \mathbb{R}^n)$  with  $m > n$ . Then for any compact  $K \subset \gamma(\bigcup_k A_k)$ , there exists an  $\hat{f} \in \text{Lip}_\ell(X, \mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} \hat{\mathbb{M}}_K(\langle T, \hat{f}, y \rangle) d\mathcal{L}^n(y) \leq \int_{\mathbb{R}^n} \hat{\mathbb{M}}_K(\langle T, f, y \rangle) d\mathcal{L}^n(y)$$

for all  $f \in \text{Lip}_\ell(X, \mathbb{R}^n)$  with

$$\partial\langle T, f, y \rangle = \partial\langle T, h, y \rangle.$$

*Proof.* First, we will denote the set of admissible functions by

$$\mathcal{A} := \{f \in \text{Lip}_\ell(X, \mathbb{R}^n) : \partial\langle T, f, y \rangle = \partial\langle T, h, y \rangle\}.$$

Notice that since  $\Omega := \cup \gamma(A_k) \subset X$  has finite Hausdorff measure, Ascoli-Arzelà gives

$\mathcal{A}$  is compact. Then if we let

$$M := \inf_{f \in \mathcal{A}} \int_{\mathbb{R}^n} \hat{\mathbb{M}}_K(\langle T, f, y \rangle) d\mathcal{L}^n(y),$$

there is a sequence  $\{f_j\} \in \mathcal{A}$  with

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} \hat{\mathbb{M}}_K(\langle T, f_j, y \rangle) d\mathcal{L}^n(y) = M,$$

which admits a subsequence, not relabeled, converging to some  $\hat{f} \in \text{Lip}(X; \mathbb{R}^n)$ :

$$f_j \rightarrow \hat{f} \text{ in } \text{Lip}(X; \mathbb{R}^n).$$

We also have convergence of the slices: applying Lemma 6.9 we find that, for almost every  $y$ ,  $\lim_{j \rightarrow \infty} \mathbb{F}_K(\langle T, f_j, y \rangle - \langle T, \hat{f}, y \rangle) = 0$ . This convergence in the flat norm implies lower semicontinuity of  $\hat{M}$ , by Theorem 6.7. Applying lower semicontinuity along with Fatou's lemma gives us the desired inequalities:

$$\begin{aligned} M &\leq \int \hat{\mathbb{M}}(\langle T, \hat{f}, y \rangle) d\mathcal{L}^n(y) \\ &\leq \int \lim_{j \rightarrow \infty} \hat{\mathbb{M}}(\langle T, f_j, y \rangle) d\mathcal{L}^n(y) \\ &\leq \liminf_{j \rightarrow \infty} \int \hat{\mathbb{M}}(\langle T, f_j, y \rangle) d\mathcal{L}^n(y) \\ &= M. \end{aligned}$$

So  $f$  is the desired minimizer.  $\square$

**Remark 6.12.** *Recall that the intuition for the above result in the Euclidean version of the theorem was the relationship given in Equation 5.5:*

$$\int_{\mathbb{R}^n} \mathbb{M}(\langle \tau(\Omega, \theta, \vec{\Omega}), f, y \rangle) d\mathcal{L}^n(y) = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(f_j^{-1}(y) \cap \Omega) d\mathcal{L}^n(y),$$

where  $\mathbb{M}$  refers to the euclidean mass. In case that  $\hat{\mathbb{M}} = \mathbb{M}$  (the metric version) and the density of  $T$  is 1 almost everywhere in its support, Eq. 6.2 provides us with the analogous statement, and we could conclude that there is a function minimizing the integral of the measure of its fibers.

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